On the monoid of cofinite partial isometries of $\mathbb{N}^n$ with the usual metric

Oleg Gutik, Anatolii Savchuk

Abstract. In this paper we study the structure of the monoid $\mathbf{IN}_\infty^n$ of cofinite partial isometries of the n -th power of the set of positive integers $\mathbb{N}$ with the usual metric for a positive integer $n \geq 2$. We describe the group of units and the subset of idempotents of the semigroup $\mathbf{IN}_\infty^n$, the natural partial order and Green's relations on $\mathbf{IN}_\infty^n$. In particular we show that the quotient semigroup $\mathbf{IN}_\infty^n / \mathfrak{C}_{\text{mg}}$, where $\mathfrak{C}_{\text{mg}}$ is the minimum group congruence on $\mathbf{IN}_\infty^n$, is isomorphic to the symmetric group $\mathcal{S}_n$ and $\mathcal{D} = \mathcal{J}$ in $\mathbf{IN}_\infty^n$. Also, we prove that for any integer $n \geq 2$ the semigroup $\mathbf{IN}_\infty^n$ is isomorphic to the semidirect product $\mathcal{S}_n \ltimes_{\mathfrak{h}} (\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ of the free semilattice with the unit $(\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ by the symmetric group $\mathcal{S}_n$.

Анотація. В роботі досліджується структура моноїда $\mathbf{IN}_\infty^n$ коскінченних часткових ізометрій n -ого степеня множини натуральних чисел $\mathbb{N}$ зі звичайною метрикою для натурального числа $n \geq 2$. Описано групу одиниць і підмножину ідемпотентів напівгрупи $\mathbf{IN}_\infty^n$, природний частковий порядок і відношення Гріна на $\mathbf{IN}_\infty^n$. Зокрема, показано, що фактор-напівгрупа $\mathbf{IN}_\infty^n / \mathfrak{C}_{\text{mg}}$, де $\mathfrak{C}_{\text{mg}}$ – найменша групова конгруенція на $\mathbf{IN}_\infty^n$, ізоморфна симетричній групі $\mathcal{S}_n$ і $\mathcal{D} = \mathcal{J}$ в $\mathbf{IN}_\infty^n$. Також, доведено, що для довільного натурального числа $n \geq 2$ напівгрупа $\mathbf{IN}_\infty^n$ ізоморфна напівпрямому добутку $\mathcal{S}_n \ltimes_{\mathfrak{h}} (\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ вільної напівґратки з одиницею $(\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ симетричною групою $\mathcal{S}_n$.

2010 Mathematics Subject Classification: 20M18, 20M20, 20M30

UDC 512.534

Keywords: Partial isometry, inverse semigroup, partial bijection, natural partial order, Green's relations, least group congruence, F -inverse semigroup, semidirect product, free semilattice, symmetric group

1. INTRODUCTION AND PRELIMINARIES

In this paper we shall follow the terminology of [3], [4], [14]. We shall denote the cardinality of the set A by $|A|$ and the first infinite cardinal by ω . For any positive integer n by $\mathcal{S}_n$ we denote the group of permutations of the set $\{1, \dots, n\}$.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of $x \in S$* . If S is an inverse semigroup, then the function $\text{inv}: S \rightarrow S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under the multiplication and we shall refer to $E(S)$ as a *band* (or the *band of S*). Then the semigroup operation on S determines the following partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents.

A linearly ordered subset of a poset is called a *chain*. An ω -*chain* is a chain which is order isomorphic to the set of all negative integers with the usual order $\leq$.

If S is an inverse semigroup then the semigroup operation on S determines the following partial order $\leq$ on S : $s \leq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [18].

By $(\mathcal{P}_\infty(X), \cup)$ we shall denote the *free semilattice with identity* over a set X of cardinality $\geq \omega$, i.e., $(\mathcal{P}_\infty(X), \cup)$ is the set of all finite subsets (including the empty set) of X with the semilattice operation “union”.

A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruences on S , and a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group. If $\mathfrak{C}$ is a congruence on a semigroup S then by $\mathfrak{C}^\#$ we denote the natural homomorphism from S onto the quotient semigroup $S/\mathfrak{C}$. Every inverse semigroup S admits the least (minimum) group congruence $\mathfrak{C}_{\text{mg}}$:

$$a\mathfrak{C}_{\text{mg}}b \text{ if and only if there exists } e \in E(S) \text{ such that } ae = be,$$

see [16, Lemma III.5.2].

If S is a semigroup, then we shall denote Green’s relations on S by $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$, see [5] or [3, Section 2.1]:

$$a\mathcal{R}b \text{ if and only if } aS^1 = bS^1;$$

$$a\mathcal{L}b \text{ if and only if } S^1a = S^1b;$$

$$\begin{aligned}
a \mathcal{J} b & \text{ if and only if } S^1 a S^1 = S^1 b S^1; \\
\mathcal{D} &= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}; \\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R}.
\end{aligned}$$

If $\alpha: X \rightarrow Y$ is a partial map, then we shall denote the domain and the range of α by $\text{dom } \alpha$ and $\text{ran } \alpha$, respectively. A partial map $\alpha: X \rightarrow Y$ is called *cofinite* if both sets $X \setminus \text{dom } \alpha$ and $Y \setminus \text{ran } \alpha$ are finite.

Let $\mathcal{J}_\lambda$ denote the set of all partial one-to-one transformations of a non-zero cardinal λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \text{ if } x \in \text{dom}(\alpha\beta) = \{y \in \text{dom } \alpha: y\alpha \in \text{dom } \beta\}, \text{ for } \alpha, \beta \in \mathcal{J}_\lambda.$$

The semigroup $\mathcal{J}_\lambda$ is called the *symmetric inverse (monoid) semigroup* over the cardinal λ , see [3]. A symmetric inverse semigroup was introduced by Wagner [18] and it plays a major role in the theory of semigroups. By $\mathcal{J}_\lambda^{\text{cf}}$ is denoted a subsemigroup of injective partial selfmaps of λ with cofinite domains and ranges in $\mathcal{J}_\lambda$. Obviously, $\mathcal{J}_\lambda^{\text{cf}}$ is an inverse submonoid of the semigroup $\mathcal{J}_\lambda$. The semigroup $\mathcal{J}_\lambda^{\text{cf}}$ is called the *monoid of injective partial cofinite selfmaps* of λ , [13].

A partial transformation $\alpha: (X, d) \rightarrow (X, d)$ of a metric space (X, d) is called *isometric* or a *partial isometry*, whenever $d(x\alpha, y\alpha) = d(x, y)$ for all $x, y \in \text{dom } \alpha$. It is obvious that the composition of two partial isometries of a metric space (X, d) is a partial isometry, and the converse partial map to a partial isometry is a partial isometry, too. Hence the set of partial isometries of a metric space (X, d) with the operation of the composition of partial isometries is an inverse submonoid of the symmetric inverse monoid over the cardinal $|X|$. Also, it is obvious that the set of partial cofinite isometries of a metric space (X, d) with the operation of the composition of partial isometries is an inverse submonoid of the monoid of injective partial cofinite selfmaps of the cardinal $|X|$.

The semigroup $\mathbf{ID}_\infty$ of all partial cofinite isometries of the set of integers $\mathbb{Z}$ with the usual metric $d(n, m) = |n - m|$, $n, m \in \mathbb{Z}$, established in Bezushchak's papers, [1, 2]. In [1] the generators of the semigroup $\mathbf{ID}_\infty$ are described and there it was proved that $\mathbf{ID}_\infty$ has the exponential growth. We remark that the semigroup $\mathbf{ID}_\infty$ is an inverse submonoid of the monoid of all partial cofinite bijections of $\mathbb{Z}$, and elements of $\mathbf{ID}_\infty$ are restrictions of isometries of $\mathbb{Z}$ onto its cofinite subsets in the Lawson interpretation, see [14, p. 9]. Green's relations and principal ideals of $\mathbf{ID}_\infty$ are described in [2]. In [9] is shown that the quotient semigroup $\mathbf{ID}_\infty / \mathfrak{C}_{\text{mg}}$ is isomorphic to the group $\mathbf{Iso}(\mathbb{Z})$ of all isometries of $\mathbb{Z}$, the semigroup $\mathbf{ID}_\infty$ is F -inverse, and $\mathbf{ID}_\infty$ is isomorphic to the semidirect product $\mathbf{Iso}(\mathbb{Z}) \ltimes_{\mathfrak{h}} \mathcal{P}_\infty(\mathbb{Z})$ of the free semilattice $(\mathcal{P}_\infty(\mathbb{Z}), \cup)$ by the group $\mathbf{Iso}(\mathbb{Z})$. Also in [9] it was established

semigroup and shift-continuous topologies on $\mathbf{ID}_\infty$ and embedding of the discrete semigroup $\mathbf{ID}_\infty$ into compact-like topological semigroups.

Let $\mathbf{IN}_\infty$ be the set of all partial cofinite isometries of the set of positive integers $\mathbb{N}$ with the usual metric $d(n, m) = |n - m|$, $n, m \in \mathbb{N}$. Then $\mathbf{IN}_\infty$ with the operation of composition of partial isometries is an inverse submonoid of $\mathcal{I}_\omega$. The semigroup $\mathbf{IN}_\infty$ of all partial co-finite isometries of positive integers is studied in [10]. There we describe Green's relations on the semigroup $\mathbf{IN}_\infty$, its band and proved that $\mathbf{IN}_\infty$ is a simple E -unitary F -inverse semigroup. Also in [10] the least group congruence $\mathfrak{C}_{\mathbf{mg}}$ on $\mathbf{IN}_\infty$ was described and it was proved that the quotient semigroup $\mathbf{IN}_\infty/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the additive group of integers $\mathbb{Z}(+)$. An example of a non-group congruence on the semigroup $\mathbf{IN}_\infty$ is presented. Also we proved that a congruence on the semigroup $\mathbf{IN}_\infty$ is group if and only if its restriction onto an isomorphic copy of the bicyclic semigroup in $\mathbf{IN}_\infty$ is a group congruence.

For an arbitrary positive integer $n \geq 2$ by $\mathbb{N}^n$ we denote the n -th power of the set of positive integers $\mathbb{N}$ with the usual metric:

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}.$$

Let $\mathbf{IN}_\infty^n$ be the set of all partial cofinite isometries of $\mathbb{N}^n$. It is obvious that $\mathbf{IN}_\infty^n$ with the operation of composition of partial isometries is an inverse submonoid of $\mathcal{I}_\omega$.

By $\mathbb{I}$ we denote the identity map of $\mathbb{N}^n$ which obviously is the unit of the semigroup $\mathbf{IN}_\infty^n$. The group of units of $\mathbf{IN}_\infty^n$ is denoted $H(\mathbb{I})$.

In the present paper we study the structure of the monoid $\mathbf{IN}_\infty^n$. We describe the group of units and the subset of idempotents of $\mathbf{IN}_\infty^n$, the natural partial order and Green's relations on $\mathbf{IN}_\infty^n$. In particular we show that the quotient semigroup $\mathbf{IN}_\infty^n/\mathfrak{C}_{\mathbf{mg}}$ is isomorphic to the symmetric group $\mathcal{S}_n$ and $\mathcal{D} = \mathcal{J}$ in $\mathbf{IN}_\infty^n$. Also, we prove that for any integer $n \geq 2$ the semigroup $\mathbf{IN}_\infty^n$ is isomorphic to the semidirect product $\mathcal{S}_n \ltimes_{\mathfrak{h}} (\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ of free semilattice with the unit $(\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ by the symmetric group $\mathcal{S}_n$.

2. PROPERTIES OF PARTIAL COFINITE ISOMETRIES OF $\mathbb{N}^n$

The definition of an isometry implies the following proposition.

Proposition 2.1. *Then every permutation σ of the set $\{1, \dots, n\}$ induces an isometry $\alpha_\sigma: \mathbb{N}^n \rightarrow \mathbb{N}^n$, $\alpha_\sigma: x \mapsto x \circ \sigma$, of the set $\mathbb{N}^n$.*

We denote $\mathbf{1} = (1, \dots, 1)$ and $\vec{\mathbf{1}}_i = \{(1, \dots, \underbrace{x_i}_{i\text{-th}}, \dots, 1) \in \mathbb{N}^n: x_i \in \mathbb{N}\}$

for any $i \in \{1, \dots, n\}$. Also for any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$ and any positive

real number r we put

$$\mathcal{M}_r(\mathbf{x}) = |\{\mathbf{y} \in \mathbb{N}^n : d(\mathbf{x}, \mathbf{y}) = r\}|.$$

Lemma 2.2. *Let n be a positive integer ≥ 2 . Then the following statements holds.*

- (i) $\mathcal{M}_1(\mathbf{1}) = n$.
- (ii) *If for $\mathbf{x} \in \mathbb{N}^n$ exactly k coordinates of $\mathbf{x}$ distinct from 1 for some $k = 1, \dots, n$, then $\mathcal{M}_1(\mathbf{x}) = n + k$.*

Proof. (i) It is obvious that the set $\{\mathbf{y} \in \mathbb{N}^n : d(\mathbf{1}, \mathbf{y}) = 1\}$ consists of

$$\{(\underbrace{2}_{1\text{-st}}, 1, \dots, 1), (1, \underbrace{2}_{2\text{-nd}}, \dots, 1), \dots, (1, \dots, \underbrace{2}_{n\text{-th}})\},$$

which implies the equality $\mathcal{M}_1(\mathbf{1}) = n$.

(ii) Fix an arbitrary $\mathbf{x} \in \mathbb{N}^n$ such that exactly k coordinates of $\mathbf{x}$ distinct from 1 for some $k \in \{1, \dots, n\}$. By Proposition 2.1 without loss of generality we may assume that only the first k coordinates of $\mathbf{x}$ distinct from 1, i.e., $\mathbf{x} = (x_1, \dots, x_k, 1, \dots, 1)$ for some $x_1 > 1, \dots, x_k > 1$. This implies that the set $\{\mathbf{y} \in \mathbb{N}^n : d(\mathbf{x}, \mathbf{y}) = 1\}$ coincides with the following set

$$\begin{aligned} &\{(x_1 - 1, \dots, x_k, 1, \dots, 1), (x_1 + 1, \dots, x_k, 1, \dots, 1), \dots, \\ &(x_1, \dots, x_k - 1, 1, \dots, 1), (x_1, \dots, x_k + 1, 1, \dots, 1), \\ &(x_1, \dots, x_k, \underbrace{2}_{(k+1)\text{-th}}, \dots, 1), \dots, (x_1, \dots, x_k, 1, \dots, \underbrace{2}_{n\text{-th}})\}, \end{aligned}$$

and hence $\mathcal{M}_1(\mathbf{x}) = n + k$. □

Statement (i) of Lemma 2.2 implies the following

Corollary 2.3. *Let n be a positive integer ≥ 2 . Then $(\mathbf{1})\alpha = \mathbf{1}$ for every isometry α of $\mathbb{N}^n$.*

For any positive integers $k > 1$ and m we denote

$$\begin{aligned} \mathbf{k}_1 &= (\underbrace{k}_{1\text{-st}}, 1, \dots, 1), \dots, \\ \mathbf{k}_j &= (1, \dots, 1, \underbrace{k}_{j\text{-th}}, 1, \dots, 1), \quad 1 < j < n, \\ \mathbf{k}_n &= (1, \dots, 1, \underbrace{k}_{n\text{-th}}) \end{aligned}$$

and

$$\mathbf{C}_m = \{(x_1, \dots, x_n) \in \mathbb{N}^n : x_1 \leq m, \dots, x_n \leq m\}.$$

Later we will need the following technical lemma.

Lemma 2.4. *Let x and y be any positive integers. If the sequence $\{a_i\}_{i \in \mathbb{N}}$, where $a_i = \sqrt{(x+i)^2 + y}$, contains an integer then the set $\{a_i\}_{i \in \mathbb{N}}$ has infinitely many non-integer elements.*

Proof. Suppose that $n = \sqrt{x^2 + y}$ is an integer. Since $x < \sqrt{x^2 + y}$ we get that

$$\begin{aligned} 2x < 2\sqrt{x^2 + y} &\iff \\ x^2 + 2x + 1 + y < x^2 + y + 2\sqrt{x^2 + y} + 1 &\iff \\ (x+1)^2 + y < (\sqrt{x^2 + y} + 1)^2 &\iff \\ \sqrt{(x+1)^2 + y} < n + 1, \end{aligned}$$

which implies that $n < \sqrt{(x+1)^2 + y} < n + 1$. Thus, $\sqrt{(x+1)^2 + y}$ is irrational, and hence the statement of the lemma holds. $\square$

Lemma 2.5. *Let $\alpha: \mathbb{N}^n \rightarrow \mathbb{N}^n$ be a partial cofinite isometry of $\mathbb{N}^n$. Then for every $i \in \{1, \dots, n\}$ there exists a unique $j(i) \in \{1, \dots, n\}$ and an integer $q(i)$ such that $(\mathbf{m}_i)\alpha = (\mathbf{m} + \mathbf{q}(\mathbf{i}))_{j(i)}$ for any $\mathbf{m}_i \in \text{dom } \alpha$. Moreover α determines the permutation $i \mapsto j(i)$ of the set $\{1, \dots, n\}$.*

Proof. Since α is a partial cofinite isometry of $\mathbb{N}^n$ there exists a positive integer $m \geq 3$ such that $\mathbb{N}^n \setminus \text{dom } \alpha \subseteq \mathbf{C}_{m-1}$ and $\mathbb{N}^n \setminus \text{ran } \alpha \subseteq \mathbf{C}_{m-1}$. Since the set $\mathbf{C}_{m+1}$ is finite there exists a positive integer k such that $\mathbf{p}_i \notin \mathbf{C}_{m+1}$ and $(\mathbf{p}_i)\alpha \notin \mathbf{C}_{m+1}$ for any integer $p \geq k$. By Proposition 2.1 without loss of generality we may assume that $i = j(i) = 1$, i.e., $(\mathbf{k}_1)\alpha = \mathbf{q}_1$ for some positive integer q .

We claim that $((\mathbf{k} + \mathbf{s})_1)\alpha = (\mathbf{q} + \mathbf{s})_1$ for any positive integer s . Since $\mathbf{k}_1 \notin \mathbf{C}_{m+1}$, we have that

$$|\{\mathbf{x} \in \text{dom } \alpha: d(\mathbf{x}, (\mathbf{k} - \mathbf{2} + \mathbf{t})_1) = 1\}| = n + 1,$$

for every positive integer t . Since α is a partial cofinite isometry of $\mathbb{N}^n$ and $(\mathbf{k} - \mathbf{1})_1 \in \text{dom } \alpha$, the above implies that either $((\mathbf{k} + \mathbf{1})_1)\alpha = (\mathbf{q} - \mathbf{1})_1$ or $((\mathbf{k} + \mathbf{1})_1)\alpha = (\mathbf{q} + \mathbf{1})_1$. We claim that $((\mathbf{k} + \mathbf{1})_1)\alpha = (\mathbf{q} + \mathbf{1})_1$. Suppose to the contrary that $((\mathbf{k} + \mathbf{1})_1)\alpha = (\mathbf{q} - \mathbf{1})_1$. Consider the following sequence $\{(\mathbf{k} + \mathbf{p})_1\}_{p \in \mathbb{N}}$ in $\mathbb{N}^n$. By the above assumption there exists a positive integer s such that $((\mathbf{k} + \mathbf{p})_1)\alpha \notin \mathbf{C}_{m+1} \cup \vec{\mathbf{1}}_1 \cup \dots \cup \vec{\mathbf{1}}_n$ for all $p \geq s$. By Lemma 2.4 the sequence $\{d(((\mathbf{k} + \mathbf{p})_1)\alpha, (\mathbf{k}_1)\alpha)\}_{p \in \mathbb{N}}$ contains non-integer numbers. But the set $\{d((\mathbf{k} + \mathbf{p})_1, \mathbf{k}_1)\}_{p \in \mathbb{N}}$ contains only positive integers, which contradicts that α is a partial isometry of $\mathbb{N}^n$. The obtained contradiction implies that $((\mathbf{k} + \mathbf{1})_1)\alpha = (\mathbf{q} + \mathbf{1})_1$. Using similar arguments by induction we get that $((\mathbf{k} + \mathbf{p})_1)\alpha = (\mathbf{q} + \mathbf{p})_1$ for any $p \in \mathbb{N}$.

Fix an arbitrary integer $l < k$ such that $\mathbf{l}_1 \in \text{dom } \alpha$. We claim that $(\mathbf{l}_1)\alpha = (\mathbf{q} - \mathbf{k} + \mathbf{l})_1$. Indeed, in the other case $d(\mathbf{l}_1, (\mathbf{k} + \mathbf{p})_1) = k + p - l$ for any $p \in \mathbb{N}$, and hence the sequence $\{d(\mathbf{l}_1, (\mathbf{k} + \mathbf{p})_1)\}_{p \in \mathbb{N}}$ contains only integer elements. But since $(\mathbf{l}_1)\alpha = (x_1, x_2, \dots, x_n)$ with some $x_i \neq 1$, $i = 2, \dots, n$, we have that

$$d((\mathbf{l}_1)\alpha, ((\mathbf{k} + \mathbf{p})_1)\alpha) = \sqrt{(x_1 - k + p)^2 + (x_2 - 1)^2 + \dots + (x_n - 1)^2},$$

and hence by Lemma 2.4 the sequence $\{d((\mathbf{l}_1)\alpha, ((\mathbf{k} + \mathbf{p})_1)\alpha)\}_{p \in \mathbb{N}}$ contains non-integer elements. This contradicts that α is a partial isometry of $\mathbb{N}^n$, which completes the proof of the lemma. $\square$

Lemma 2.6. *Let n be a positive integer ≥ 2 . Let $\alpha: \mathbb{N}^n \rightarrow \mathbb{N}^n$ be a partial cofinite isometry of $\mathbb{N}^n$ such that $(\mathbf{m}_i)\alpha \in \vec{\mathbf{l}}_i$ for some $\mathbf{m}_i \in \text{dom } \alpha \setminus \{\mathbf{1}\}$, $m \in \mathbb{N}$, and any $i \in \{1, \dots, n\}$. Then $(\mathbf{x}_i)\alpha = \mathbf{x}_i$ for any $\mathbf{x}_i \in \text{dom } \alpha \cap \vec{\mathbf{l}}_i$.*

Proof. Suppose to the contrary that there exists $i \in \{1, \dots, n\}$ such that $(\mathbf{x}_i)\alpha \neq \mathbf{x}_i$ for some $\mathbf{x}_i \in \vec{\mathbf{l}}_i$. Without loss of generality we may assume that $i = 1$. Then there exists a non-zero integer p such that $(\mathbf{x}_1)\alpha = (\mathbf{x} + \mathbf{p})_1$.

First consider the case $p > 0$. Fix an arbitrary $\mathbf{y}_2 \in \text{dom } \alpha \cap \vec{\mathbf{l}}_2 \setminus \{\mathbf{1}\}$. Lemma 2.5 implies that without loss of generality we may assume that $y < x$. Since α is a partial isometry of $\mathbb{N}^n$,

$$d((\mathbf{x}_1)\alpha, (\mathbf{y}_2)\alpha) = d(\mathbf{x}_1, \mathbf{y}_2) = \sqrt{(x - 1)^2 + (y - 1)^2}. \quad (2.1)$$

Since $(\mathbf{x}_1)\alpha = (\mathbf{x} + \mathbf{p})_1$ equality (2.1) implies that there exists a positive integer q such that $(\mathbf{y}_2)\alpha = (\mathbf{x} - \mathbf{q})_2$. Then we have that

$$\begin{aligned} d((\mathbf{x}_1)\alpha, (\mathbf{y}_2)\alpha) &= d((\mathbf{x} + \mathbf{p})_1, (\mathbf{y} - \mathbf{q})_2) = \\ &= \sqrt{(x + p - 1)^2 + (y - q - 1)^2}. \end{aligned} \quad (2.2)$$

By equalities (2.1) and (2.2) we get that

$$(x - 1)^2 + (y - 1)^2 = (x + p - 1)^2 + (y - q - 1)^2,$$

and hence

$$p^2 + q^2 + 2(xp - yq) - 2(p - q) = 0. \quad (2.3)$$

Again, since α is a partial isometry of $\mathbb{N}^n$, Lemma 2.5 implies that

$$\begin{aligned} d(((\mathbf{x} + \mathbf{1})_1)\alpha, ((\mathbf{y} + \mathbf{1})_2)\alpha) &= d((\mathbf{x} + \mathbf{1})_1, (\mathbf{y} + \mathbf{1})_2) = \\ &= \sqrt{(x + 1 - 1)^2 + (y + 1 - 1)^2} = \sqrt{x^2 + y^2}, \end{aligned} \quad (2.4)$$

$((\mathbf{x} + \mathbf{1})_1)\alpha = (\mathbf{x} + \mathbf{1} + \mathbf{p})_1$ and $((\mathbf{y} + \mathbf{1})_2)\alpha = (\mathbf{y} + \mathbf{1} - \mathbf{q})_2$. Then

$$\begin{aligned} d(((\mathbf{x} + \mathbf{1})_1)\alpha, ((\mathbf{y} + \mathbf{1})_2)\alpha) &= d((\mathbf{x} + \mathbf{1} + \mathbf{p})_1, (\mathbf{y} + \mathbf{1} - \mathbf{q})_2) = \\ &= \sqrt{(x + 1 + p - 1)^2 + (y + 1 - q - 1)^2} = \\ &= \sqrt{(x + p)^2 + (y - q)^2}. \end{aligned} \quad (2.5)$$

By equalities (2.4) and (2.5) we have that

$$x^2 + y^2 = x^2 + 2xp + p^2 + y^2 - 2yq + q^2,$$

and after some arithmetic simplifications we get that

$$p^2 + q^2 + 2(xp - yq) = 0. \quad (2.6)$$

Then equalities (2.3) and (2.6) imply that $p = q$. Therefore,

$$(\mathbf{y}_2)\alpha = (\mathbf{x} - \mathbf{p})_2.$$

Hence due to (2.1) and (2.2)

$$(x - 1)^2 + (y - 1)^2 = (x - 1)^2 + 2(x - 1)p + p^2 + (y - 1)^2 - 2(y - 1)p + p^2,$$

and so

$$2p(x - y + p) = 0.$$

This contradicts the assumptions $p > 0$ and $x > y$.

If $p < 0$ choose $x < y$. Similarly as above we get a contradiction.

The obtained contradictions imply that $(\mathbf{x}_i)\alpha = \mathbf{x}_i$ for any $\mathbf{x}_i \in \vec{\mathbf{I}}_i$ with $x \geq k$, $i = 1, \dots, n$. This completes the proof of the lemma. $\square$

Theorem 2.7. *Let $\alpha: \mathbb{N}^n \rightarrow \mathbb{N}^n$ be a partial cofinite isometry of $\mathbb{N}^n$ such that $(\mathbf{m}_i)\alpha \in \vec{\mathbf{I}}_i$ for some $\mathbf{m}_i \in \text{dom } \alpha \setminus \{\mathbf{1}\}$, $m \in \mathbb{N}$, and any $i \in \{1, \dots, n\}$. Then $(\mathbf{x})\alpha = \mathbf{x}$ for any $\mathbf{x} \in \text{dom } \alpha$.*

Proof. By Lemma 2.6 it is sufficiently to show that $(\mathbf{x})\alpha = \mathbf{x}$ for any $\mathbf{x} \in \text{dom } \alpha \setminus (\vec{\mathbf{I}}_1 \cup \dots \cup \vec{\mathbf{I}}_n)$.

Suppose to the contrary that there exists

$$\mathbf{x} = (x_1, \dots, x_n) \in \text{dom } \alpha \setminus (\vec{\mathbf{I}}_1 \cup \dots \cup \vec{\mathbf{I}}_n)$$

such that $x_1, \dots, x_n \neq 1$ and $(\mathbf{x})\alpha = (y_1, \dots, y_n) \neq \mathbf{x}$. Lemma 2.5 implies that $(\mathbf{p}_1)\alpha = \mathbf{p}_1$ for all positive integers p such that $\mathbf{p}_1 \in \text{dom } \alpha$. Then we have that

$$\begin{aligned} d(\mathbf{x}, \mathbf{p}_1) &= d((\mathbf{x})\alpha, (\mathbf{p}_1)\alpha) \iff \\ (x_1 - p)^2 + (x_2 - 1)^2 + \dots + (x_n - 1)^2 &= (y_1 - p)^2 + (y_2 - 1)^2 + \dots + (y_n - 1)^2 \iff \\ x_1^2 - 2x_1p + p^2 + (x_2 - 1)^2 + \dots + (x_n - 1)^2 - \\ - y_1^2 + 2y_1p - p^2 - (y_2 - 1)^2 - \dots - (y_n - 1)^2 &= 0 \iff \end{aligned}$$

$$2(y_1 - x_1)p + (x_2 - 1)^2 + \cdots + (x_n - 1)^2 + x_1^2 - y_1^2 - (y_2 - 1)^2 - \cdots - (y_n - 1)^2 = 0. \quad (2.7)$$

Since the partial map α is cofinite, equation (2.7) has infinitely many solutions in $\mathbb{N}$ with the respect to the variable p . This implies that $y_1 = x_1$ and

$$(x_2 - 1)^2 + \cdots + (x_n - 1)^2 + x_1^2 - y_1^2 - (y_2 - 1)^2 - \cdots - (y_n - 1)^2 = 0.$$

Similar arguments and Lemma 2.5 imply that $y_i = x_i$ for any other $i \in \{2, \dots, n\}$. $\square$

3. ALGEBRAIC PROPERTY OF THE SEMIGROUP $\mathbf{IN}_\infty^n$

Theorem 2.7 implies the following corollary.

Corollary 3.1. *Let n be any positive integer ≥ 2 and $\alpha: \mathbb{N}^n \rightarrow \mathbb{N}^n$ be an isometry such that $(\mathbf{2}_i)\alpha = \mathbf{2}_i$ for all $i = 1, \dots, n$. Then α is the identity map of $\mathbb{N}^n$.*

The following theorem describes the structure of the group of units $H(\mathbb{I})$ of the semigroup $\mathbf{IN}_\infty^n$.

Theorem 3.2. *The group of units $H(\mathbb{I})$ of the semigroup $\mathbf{IN}_\infty^n$ is isomorphic to the group $\mathcal{S}_n$.*

Proof. It is obvious that every element of the group of units $H(\mathbb{I})$ of the semigroup $\mathbf{IN}_\infty^n$ is an isometry of $\mathbb{N}^n$.

Fix an arbitrary isometry α of $\mathbb{N}^n$. Since the set $\{\mathbf{y} \in \mathbb{N}^n: d(\mathbf{1}, \mathbf{y}) = 1\}$ coincides with

$$\mathbf{D}_1(\mathbf{1}) = \{(\underbrace{2}_{1\text{-st}}, 1, \dots, 1), (1, \underbrace{2}_{2\text{-nd}}, \dots, 1), \dots, (1, \dots, \underbrace{2}_{n\text{-th}})\},$$

we have that $(\mathbf{D}_1(\mathbf{1}))\alpha = \mathbf{D}_1(\mathbf{1})$. Then there exists a permutation $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that $(\mathbf{2}_i)\alpha = \mathbf{2}_{(i)\sigma}$ for any $i = 1, \dots, n$. By Proposition 2.1 the permutation σ induces the isometry $\alpha_\sigma: \mathbb{N}^n \rightarrow \mathbb{N}^n$, $x \mapsto x \circ \sigma$ of the set $\mathbb{N}^n$. It is obvious that $(\mathbf{2}_i)\alpha\alpha_\sigma^{-1} = \mathbf{2}_i$ and $(\mathbf{2}_j)\alpha_\sigma^{-1}\alpha = \mathbf{2}_j$ for all $i, j = 1, \dots, n$. Since $\alpha\alpha_\sigma^{-1}$ and $\alpha_\sigma^{-1}\alpha$ are isometries of $\mathbb{N}^n$, Corollary 3.1 implies that $\alpha\alpha_\sigma^{-1} = \alpha_\sigma^{-1}\alpha = \mathbb{I}$. Hence $\alpha = \alpha_\sigma$. It is obvious that every element σ of the group $\mathcal{S}_n$ act on $\mathbb{N}^n$ in the following way:

$$(x_1, \dots, x_n)\sigma = (x_{(1)\sigma^{-1}}, \dots, x_{(n)\sigma^{-1}}), \quad (x_1, \dots, x_n) \in \mathbb{N}^n.$$

This completes the proof of the theorem. $\square$

Proposition 3.3. *The following statements hold.*

- (i) An element α of the semigroup $\mathbf{IN}_\infty^n$ is an idempotent if and only if $(x)\alpha = x$ for every $x \in \text{dom } \alpha$.
- (ii) If $\varepsilon, \iota \in E(\mathbf{IN}_\infty^n)$, then $\varepsilon \leq \iota$ if and only if $\text{dom } \varepsilon \subseteq \text{dom } \iota$.
- (iii) The semilattice $E(\mathbf{IN}_\infty^n)$ is isomorphic to the free semilattice $(\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ under the mapping $(\varepsilon)h = \mathbb{N} \setminus \text{dom } \varepsilon$.
- (iv) Every maximal chain in $E(\mathbf{IN}_\infty^n)$ is an ω -chain.
- (v) $\mathbf{IN}_\infty^n$ is an inverse semigroup.
- (vi) $\alpha \mathcal{R} \beta$ in $\mathbf{IN}_\infty^n$ if and only if $\text{dom } \alpha = \text{dom } \beta$.
- (vii) $\alpha \mathcal{L} \beta$ in $\mathbf{IN}_\infty^n$ if and only if $\text{ran } \alpha = \text{ran } \beta$.
- (viii) $\alpha \mathcal{H} \beta$ in $\mathbf{IN}_\infty^n$ if and only if $\text{dom } \alpha = \text{dom } \beta$ and $\text{ran } \alpha = \text{ran } \beta$.

Proof. Statements (i)-(iv) are trivial and their proofs follow from the definition of the semigroup $\mathbf{IN}_\infty^n$.

(v) Fix an arbitrary $\alpha \in \mathbf{IN}_\infty^n$. Since α is a partial cofinite isometry of $\mathbb{N}^n$, we conclude that so is its inverse α^{-1} . This implies that $\mathbf{IN}_\infty^n$ is regular, and hence by statement (iii) and the Wagner-Preston Theorem, see [3, Theorem 1.17], $\mathbf{IN}_\infty^n$ is an inverse semigroup.

Statements (vi)-(viii) follow from the description of Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ on the symmetric inverse monoid $\mathcal{I}_\omega$ and [14, Proposition 3.2.11]. $\square$

Lemma 3.4. *Let α be an arbitrary element of the semigroup $\mathbf{IN}_\infty^n$. Then there exist unique elements σ_l and σ_r of the group of units $H(\mathbb{I})$ of the semigroup $\mathbf{IN}_\infty^n$ such that $\sigma_l \alpha$ and $\alpha \sigma_r$ are idempotents in $\mathbf{IN}_\infty^n$.*

Proof. Suppose that $\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, $i \mapsto \sigma(i)$ is a permutation defined in Lemma 2.5 for the partial isometry α . We put α_σ to be an element of $H(\mathbb{I})$ which is induced by the permutation σ of the set $\{1, \dots, n\}$, see Theorem 3.2, and α_σ^{-1} is its inverse. Then the assumption of Theorem 2.7 holds for $\alpha \alpha_\sigma^{-1}$, and hence by Proposition 3.3(i), $\alpha \alpha_\sigma^{-1}$ is an idempotent of $\mathbf{IN}_\infty^n$. Next we put $\sigma_r = \alpha_\sigma^{-1}$.

Let $\bar{\alpha}_\sigma$ be the element of $H(\mathbb{I})$ which is induced by the permutation σ^{-1} of the set $\{1, \dots, n\}$. Then by Proposition 3.3(i) the element $\alpha^{-1} \bar{\alpha}_\sigma^{-1}$ is an idempotent of $\mathbf{IN}_\infty^n$, and hence so is its inverse $(\alpha^{-1} \bar{\alpha}_\sigma^{-1})^{-1} = \bar{\alpha}_\sigma \alpha$, because by Proposition 3.3(v) the semigroup $\mathbf{IN}_\infty^n$ is inverse. Hence, we put $\sigma_l = \alpha_\sigma$.

Observe finally, that the uniqueness of the elements σ_l and σ_r follows from Lemma 2.5. $\square$

Lemma 3.5. *Let α be an arbitrary element of the semigroup $\mathbf{IN}_\infty^n$. Then there exists the unique element α_σ of the group of units $H(\mathbb{I})$ and the unique*

idempotents $\varepsilon_{l(\alpha)}$ and $\varepsilon_{r(\alpha)}$ of the semigroup $\mathbf{IN}_\infty^n$ such that

$$\alpha = \alpha_\sigma \varepsilon_{l(\alpha)} = \varepsilon_{r(\alpha)} \alpha_\sigma.$$

Proof. By Lemma 3.4 there exist unique elements σ_{l_α} and σ_{r_α} of the group of units $H(\mathbb{I})$ such that $\sigma_{l_\alpha} \alpha$ and $\alpha \sigma_{r_\alpha}$ are idempotents of the semigroup $\mathbf{IN}_\infty^n$. Since $\mathbf{IN}_\infty^n$ is an inverse semigroup, $\text{ran}(\sigma_{l_\alpha} \alpha) = \text{ran}(\alpha^{-1} \alpha)$ and $\text{dom}(\alpha \sigma_{r_\alpha}) = \text{dom}(\alpha \alpha^{-1})$ above imply that

$$\sigma_{l_\alpha} \alpha = \alpha^{-1} \alpha \quad \text{and} \quad \alpha \sigma_{r_\alpha} = \alpha \alpha^{-1}. \quad (3.1)$$

Hence we get that

$$\alpha = \sigma_{l_\alpha} \sigma_{l_\alpha} \alpha = \sigma_{l_\alpha} \alpha^{-1} \alpha \quad \text{and} \quad \alpha = \alpha \sigma_{r_\alpha} \sigma_{r_\alpha}^{-1} = \alpha \alpha^{-1} \sigma_{r_\alpha}^{-1}.$$

Put

$$\sigma_{l(\alpha)} = \sigma_{l_\alpha}, \quad \sigma_{r(\alpha)} = \sigma_{r_\alpha}^{-1}, \quad \varepsilon_{l(\alpha)} = \alpha^{-1} \alpha, \quad \varepsilon_{r(\alpha)} = \alpha \alpha^{-1}.$$

We claim that $\sigma_{l(\alpha)} = \sigma_{r(\alpha)}$. Indeed, the inequality $\sigma_{l(\alpha)} \neq \sigma_{r(\alpha)}$ and Theorem 3.2 imply that the restriction $\sigma_{l(\alpha)}^{-1} \sigma_{r(\alpha)}|_{\mathbb{N}^n \setminus \mathbf{C}_m}$ is not the identity map for any positive integer m . Hence $\varepsilon_{l(\alpha)} \neq \sigma_{l(\alpha)}^{-1} \varepsilon_{r(\alpha)} \sigma_{r(\alpha)}$, which contradicts the equalities $\alpha = \sigma_{l(\alpha)} \varepsilon_{l(\alpha)} = \varepsilon_{r(\alpha)} \sigma_{r(\alpha)}$. Therefore $\sigma_{l(\alpha)} = \sigma_{r(\alpha)}$ and we put $\alpha_\sigma = \sigma_{r(\alpha)}$.

The uniqueness of the elements α_σ , $\varepsilon_{l(\alpha)}$ and $\varepsilon_{r(\alpha)}$ follows from Lemma 2.5 and equalities (3.1), because $\mathbf{IN}_\infty^n$ is an inverse semigroup. $\square$

Lemma 3.5 implies the following corollary.

Corollary 3.6. *For any positive integer $n \geq 2$ the subset*

$$H(\mathbb{I}) \cup E(\mathbf{IN}_\infty^n) \subset \mathbf{IN}_\infty^n$$

generates the semigroup $\mathbf{IN}_\infty^n$.

The following corollary describes the natural partial order on the semigroups $\mathbf{IN}_\infty^n$ in the terms of Lemma 3.5.

Corollary 3.7. *Let n be any positive integer ≥ 2 , and α and β be elements of the semigroup $\mathbf{IN}_\infty^n$ such that*

$$\alpha = \alpha_\sigma \varepsilon_{l(\alpha)} = \varepsilon_{r(\alpha)} \alpha_\sigma \quad \text{and} \quad \beta = \beta_\sigma \varepsilon_{l(\beta)} = \varepsilon_{r(\beta)} \beta_\sigma$$

for some elements α_σ and β_σ of the group of units $H(\mathbb{I})$ and idempotents $\varepsilon_{l(\alpha)}$, $\varepsilon_{r(\alpha)}$, $\varepsilon_{l(\beta)}$ and $\varepsilon_{r(\beta)}$ of the semigroup $\mathbf{IN}_\infty^n$. Then $\alpha \leq \beta$ in $\mathbf{IN}_\infty^n$ if and only if $\alpha_\sigma = \beta_\sigma$, $\varepsilon_{l(\alpha)} \leq \varepsilon_{l(\beta)}$ and $\varepsilon_{r(\alpha)} \leq \varepsilon_{r(\beta)}$ in $E(\mathbf{IN}_\infty^n)$.

Since $\alpha \mathfrak{C}_{\text{mg}} \beta$ in $\mathbf{IN}_\infty^n$ if and only if there exists $\gamma \in \mathbf{IN}_\infty^n$ such that $\gamma \leq \alpha$ and $\gamma \leq \beta$, see [14, Section 2.4, p. 62], Corollary 3.7 implies that $\alpha \mathfrak{C}_{\text{mg}} \beta$ in $\mathbf{IN}_\infty^n$ if and only if $\alpha_\sigma = \beta_\sigma$. Hence the following theorem holds.

Theorem 3.8. *Let n be any positive integer ≥ 2 . Then the quotient semigroup $\mathbb{IN}_\infty^n / \mathfrak{C}_{mg}$ is isomorphic to the group $\mathcal{S}_n$ and the natural homomorphism $\mathfrak{C}_{mg}^\# : \mathbb{IN}_\infty^n \rightarrow \mathbb{IN}_\infty^n / \mathfrak{C}_{mg}$ is defined by $\alpha \mapsto \alpha_\sigma$.*

The following theorem gives a more detailed description of Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on the semigroup $\mathbb{IN}_\infty^n$.

Theorem 3.9. *Let n be any positive integer ≥ 2 and $\alpha, \beta \in \mathbb{IN}_\infty^n$. Then the following statements hold:*

- (i) $\alpha \mathcal{L} \beta$ if and only if there exists $\sigma \in H(\mathbb{I})$ such that $\alpha = \sigma \beta$;
- (ii) $\alpha \mathcal{R} \beta$ if and only if there exists $\sigma \in H(\mathbb{I})$ such that $\alpha = \beta \sigma$;
- (iii) $\alpha \mathcal{H} \beta$ if and only if there exist $\sigma_1, \sigma_2 \in H(\mathbb{I})$ such that $\alpha = \sigma_1 \beta$ and $\alpha = \beta \sigma_2$;
- (iv) $\alpha \mathcal{D} \beta$ if and only if there exist $\sigma_1, \sigma_2 \in H(\mathbb{I})$ such that $\alpha = \sigma_1 \beta \sigma_2$;
- (v) $\mathcal{D} = \mathcal{J}$ on $\mathbb{IN}_\infty^n$;
- (vi) every $\mathcal{J}$ -class in $\mathbb{IN}_\infty^n$ is finite and consists of incomparable elements with the respect to the natural partial order $\leq$ on $\mathbb{IN}_\infty^n$.

Proof. (i) ($\Rightarrow$) Suppose that $\alpha \mathcal{L} \beta$ in $\mathbb{IN}_\infty^n$. Then $\alpha^{-1}\alpha = \beta^{-1}\beta$ and by Lemma 3.4 there exist unique elements σ_{l_α} and σ_{l_β} of the group of units $H(\mathbb{I})$ such that $\sigma_{l_\alpha}\alpha$ and $\sigma_{l_\beta}\alpha$ are idempotents of the semigroup $\mathbb{IN}_\infty^n$. This implies that

$$\sigma_{l_\alpha}\alpha = \alpha^{-1}\alpha = \beta^{-1}\beta = \sigma_{l_\beta}\beta,$$

because $\mathbb{IN}_\infty^n$ is an inverse semigroup and

$$\text{ran}(\sigma_{l_\alpha}\alpha) = \text{ran}(\alpha^{-1}\alpha) = \text{ran}(\beta^{-1}\beta) = \text{ran}(\sigma_{l_\beta}\beta).$$

Hence $\alpha = \mathbb{I}\alpha = \sigma_{l_\alpha}^{-1}\sigma_{l_\alpha}\alpha = \sigma_{l_\alpha}^{-1}\sigma_{l_\beta}\beta$, and $\sigma = \sigma_{l_\alpha}^{-1}\sigma_{l_\beta} \in H(\mathbb{I})$ is the required element.

($\Leftarrow$) Suppose that there exists $\sigma \in H(\mathbb{I})$ such that $\alpha = \sigma \beta$. Then

$$\alpha^{-1}\alpha = (\sigma \beta)^{-1}\sigma \beta = \beta^{-1}\sigma^{-1}\sigma \beta = \beta^{-1}\mathbb{I}\beta = \beta^{-1}\beta,$$

and hence $\alpha \mathcal{L} \beta$ in $\mathbb{IN}_\infty^n$.

The proof of statement (ii) is similar to (i).

Statement (iii) follows from (i) and (ii).

(iv) ($\Rightarrow$) Suppose that $\alpha \mathcal{D} \beta$ in $\mathbb{IN}_\infty^n$. Then there exists $\gamma \in \mathbb{IN}_\infty^n$ such that $\alpha \mathcal{L} \gamma$ and $\gamma \mathcal{R} \beta$. Hence by statements (i) and (ii) we get that there exist $\sigma_1, \sigma_2 \in H(\mathbb{I})$ such that $\alpha = \sigma_1 \gamma$ and $\gamma = \beta \sigma_2$. Therefore $\alpha = \sigma_1 \beta \sigma_2$.

($\Leftarrow$) Suppose that there exist $\sigma_1, \sigma_2 \in H(\mathbb{I})$ such that $\alpha = \sigma_1 \beta \sigma_2$. Then $\alpha = \sigma_1 \gamma$ for $\gamma = \beta \sigma_2$, whence $\alpha \mathcal{L} \gamma$ and $\gamma \mathcal{R} \beta$ due to (i) and (ii). Therefore $\alpha \mathcal{D} \beta$ in $\mathbb{IN}_\infty^n$.

(v) By [14, Proposition 3.2.17] it suffices to show that every $\mathcal{D}$ -class D in $\mathbb{IN}_\infty^n$ contains a minimal element with the respect to the natural partial order $\leq$ on $\mathbb{IN}_\infty^n$. Fix an arbitrary $\mathcal{D}$ -class D in $\mathbb{IN}_\infty^n$ and any $\alpha \in D$. Then by Lemma 3.5 there exists a unique element σ_α of the group of units $H(\mathbb{I})$ and a unique idempotent $\varepsilon_{l(\alpha)}$ of $\mathbb{IN}_\infty^n$ such that $\alpha = \sigma_\alpha \varepsilon_{l(\alpha)}$. Then $\sigma_\alpha^{-1} \alpha = \sigma_\alpha^{-1} \sigma_\alpha \varepsilon_{l(\alpha)} = \mathbb{I} \varepsilon_{l(\alpha)} = \varepsilon_{l(\alpha)}$ and $\varepsilon_{l(\alpha)} \in D$ by statement (iv). Also statement (iv) and Theorem 3.2 imply that if an idempotent ε belongs to D , then $|\mathbb{N}^n \setminus \text{dom } \varepsilon| = |\mathbb{N}^n \setminus \text{dom } \varepsilon_{l(\alpha)}|$, and hence all idempotents in $\mathcal{D}$ -class D are incomparable with respect to the natural partial order $\leq$ on $\mathbb{IN}_\infty^n$. Thus, every $\mathcal{D}$ -class D in $\mathbb{IN}_\infty^n$ contains a minimal element.

(vi) The first part of the statement follows from (v), (iv), and Theorem 3.2.

Let α and β be elements belonging to some $\mathcal{J}$ -class j in $\mathbb{IN}_\infty^n$ such that $\alpha \leq \beta$. Then by Lemma 3.5 there exist unique elements $\sigma_\alpha, \sigma_\beta \in H(\mathbb{I})$ and $\varepsilon_{l(\alpha)}, \varepsilon_{l(\beta)} \in E(\mathbb{IN}_\infty^n)$ such that $\alpha = \sigma_\alpha \varepsilon_{l(\alpha)}$ and $\beta = \sigma_\beta \varepsilon_{l(\beta)}$. By Corollary 3.7, $\sigma_\alpha = \sigma_\beta$ and $\varepsilon_{l(\alpha)} \leq \varepsilon_{l(\beta)}$ in $E(\mathbb{IN}_\infty^n)$. Then

$$\sigma_\alpha^{-1} \alpha = \sigma_\alpha^{-1} \sigma_\alpha \varepsilon_{l(\alpha)} = \varepsilon_{l(\alpha)} \quad \text{and} \quad \sigma_\alpha^{-1} \beta = \sigma_\alpha^{-1} \sigma_\beta \varepsilon_{l(\beta)} = \varepsilon_{l(\beta)}$$

are $\mathcal{J}$ -equivalent idempotents in $\mathbb{IN}_\infty^n$. By the proof of statement (v) we get that $\varepsilon_{l(\alpha)} = \varepsilon_{l(\beta)}$, and hence by Lemma 3.5, $\alpha = \beta$. $\square$

4. THE STRUCTURE THEOREM FOR THE SEMIGROUP $\mathbb{IN}_\infty^n$

Following [15], an inverse semigroup S is said to be *F-inverse* if every $\mathfrak{C}_{\text{mg}}$ -class has a maximum element. It is obvious that every *F-inverse* semigroup contains a unit. An inverse semigroup S is *E-unitary* if, whenever e is an idempotent in S , $s \in S$ and $e \leq s$, then s is an idempotent in S , [17].

Lemma 3.5 implies the following two corollaries.

Corollary 4.1. $\mathbb{IN}_\infty^n$ is an *F-inverse* semigroup for any positive integer $n \geq 2$.

Corollary 4.2. $\mathbb{IN}_\infty^n$ is an *E-unitary* inverse semigroup for any positive integer $n \geq 2$.

For any element s of an inverse semigroup S we denote $\downarrow s = \{x \in S : x \leq s\}$, where $\leq$ is the natural partial order on S .

Let S be any *F-inverse* semigroup. Then for any $s \in S$ by e_s we denote the idempotent $ss^{-1} \in S$ and by t_s the maximum element in the $\mathfrak{C}_{\text{mg}}$ -class of $s \in S$, and by T_S the set $\{t_s : s \in S\}$. Notice that T is the set of maximal elements of S , and that T_S need not be closed under the multiplication, [15].

The structure of F -inverse semigroups is described in [15], and later we will need the following two statements for the description of the structure of the semigroup $\mathbf{IN}_\infty^n$.

Lemma 4.3. [15, Lemma 3] *Let S be an F -inverse semigroup and 1_S denote the identity element of S . Then*

- (i) $E(S)$ is a semilattice with 1_S for an identity element.
- (ii) On T_S define the following multiplication:

$$u * v = t_{uv}, \quad \text{for } u, v \in T_S.$$

Then $(T_S, *)$ is a group with 1_S for an identity element, and each $t \in T_S$ has t^{-1} as its group inverse in $(T_S, *)$.

- (iii) For each $t \in T_S$ the map $\mathfrak{F}_t: E(S) \rightarrow \downarrow e_t$, defined by

$$(f)\mathfrak{F}_t = tft^{-1}, \quad f \in E(S),$$

is a homomorphism onto $\downarrow e_t$, and moreover $\mathfrak{F}_{1_S}$ is the identity map on $E(S)$.

- (iv) For each $t \in T_S$ $(1_S)\mathfrak{F}_t = e_t$ and $(e_t)\mathfrak{F}_{t^{-1}} = e_{t^{-1}}$.

- (v) For each $u, v \in T_S$

$$((1_S)\mathfrak{F}_u)\mathfrak{F}_v \cdot (f)\mathfrak{F}_{u*v} = ((f)\mathfrak{F}_u)\mathfrak{F}_v, \quad \text{for all } f \in E(S);$$

- (vi) If $u, v \in T_S$ then

$$f \cdot (g)\mathfrak{F}_u \leq e_{u*v},$$

for all idempotents $f \leq e_u$ and $g \leq e_v$ of S .

Theorem 4.4. [15, Theorem 3] *Let S be an F -inverse semigroup and let $\mathcal{S} = \bigcup_{t \in T_S} (\downarrow e_t \times \{t\})$. Define the multiplication $\circ$ on $\mathcal{S}$ as follows. If $u, v \in T_S$ then for idempotents $f \leq e_u$ and $g \leq e_v$ put*

$$(f, u) \circ (g, v) = (f \cdot (g)\mathfrak{F}_u, u * v). \quad (4.1)$$

Then $\circ$ is a well-defined multiplication on $\mathcal{S}$ and $(\mathcal{S}, \circ)$ is isomorphic to S under the map $\mathfrak{H}: S \rightarrow \mathcal{S}, s \mapsto (ss^{-1}, t_s)$.

Let A and B be semigroups, $\text{End}(B)$ be the semigroup of endomorphisms of B , and $\mathfrak{h}: A \rightarrow \text{End}(B): b \mapsto \mathfrak{h}_b$ be a homomorphism. Then the set $A \times B$ with the semigroup operation

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, (b_1)\mathfrak{h}_{a_2} b_2), \quad a_1, a_2 \in A, b_1, b_2 \in B,$$

is called the *semidirect product* of the semigroup A by B with the respect to the homomorphism $\mathfrak{h}$ and it is denoted by $A \ltimes_{\mathfrak{h}} B$ [14]. In this case we say that the right action of the semigroup A is defined on the semigroup

of endomorphisms of B . Notice that a semidirect product of two inverse semigroups is not need to be an inverse semigroup, see [14, Section 5.3].

Lemma 4.5. *Let $\mathfrak{h}: H(\mathbb{I}) \rightarrow \text{End}(E(\mathbb{I}\mathbb{N}_\infty^n))$ be the map defined by the formula $\sigma \mapsto \mathfrak{h}_\sigma$, where $(\alpha)\mathfrak{h}_\sigma = \sigma^{-1}\alpha\sigma$ is the automorphism of the semilattice $E(\mathbb{I}\mathbb{N}_\infty^n)$. Then $\mathfrak{h}$ is a homomorphism, and moreover $\mathfrak{h}_1$ is the identity automorphism of $E(\mathbb{I}\mathbb{N}_\infty^n)$.*

Proof. For any $\sigma \in H(\mathbb{I})$, $\varepsilon, \iota \in E(\mathbb{I}\mathbb{N}_\infty^n)$ we have that

$$(\varepsilon\iota)\mathfrak{h}_\sigma = \sigma^{-1}\varepsilon\iota\sigma = \sigma^{-1}\varepsilon\sigma\sigma^{-1}\iota\sigma = (\varepsilon)\mathfrak{h}_\sigma(\iota)\mathfrak{h}_\sigma,$$

and hence $\mathfrak{h}_\sigma$ is an endomorphism of the semilattice $E(\mathbb{I}\mathbb{N}_\infty^n)$. Also, since for any $\sigma \in H(\mathbb{I})$ and $\varepsilon \in E(\mathbb{I}\mathbb{N}_\infty^n)$ the element $\sigma\varepsilon\sigma^{-1}$ is an idempotent of the semigroup $\mathbb{I}\mathbb{N}_\infty^n$ and $(\sigma\varepsilon\sigma^{-1})\mathfrak{h}_\sigma = \sigma^{-1}(\sigma\varepsilon\sigma^{-1})\sigma = \varepsilon$, the homomorphism $\mathfrak{h}_\sigma$ is a surjective map. It is obvious that $\mathfrak{h}_1$ is the identity automorphism of the semilattice $E(\mathbb{I}\mathbb{N}_\infty^n)$.

Suppose that $(\varepsilon)\mathfrak{h}_\sigma = (\iota)\mathfrak{h}_\sigma$ for some $\gamma \in H(\mathbb{I})$ and $\varepsilon, \iota \in E(\mathbb{I}\mathbb{N}_\infty^n)$. Since $H(\mathbb{I})$ is the group of units of the semigroup $\mathbb{I}\mathbb{N}_\infty^n$, the following equalities

$$\sigma^{-1}\varepsilon\sigma = (\varepsilon)\mathfrak{h}_\sigma = (\iota)\mathfrak{h}_\sigma = \sigma^{-1}\iota\sigma$$

imply that

$$\varepsilon = 1\varepsilon 1 = \sigma\sigma^{-1}\varepsilon\sigma\sigma^{-1} = \sigma\sigma^{-1}\iota\sigma\sigma^{-1} = 1\iota 1 = \iota,$$

and hence $\mathfrak{h}_\sigma$ is an automorphism of the semilattice $E(\mathbb{I}\mathbb{N}_\infty^n)$.

Fix arbitrary $\sigma_1, \sigma_2 \in H(\mathbb{I})$. Then for any idempotent $\varepsilon \in \mathbb{I}\mathbb{N}_\infty^n$ we have that

$$\begin{aligned} (\varepsilon)\mathfrak{h}_{\sigma_1\sigma_2} &= (\sigma_1\sigma_2)^{-1}\varepsilon\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}\varepsilon\sigma_1\sigma_2 = \sigma_2^{-1}(\varepsilon)\mathfrak{h}_{\sigma_1}\sigma_2 = \\ &= ((\varepsilon)\mathfrak{h}_{\sigma_1})\mathfrak{h}_{\sigma_2} = (\varepsilon)(\mathfrak{h}_{\sigma_1} \cdot \mathfrak{h}_{\sigma_2}), \end{aligned}$$

and hence so defined map $\mathfrak{h}: H(\mathbb{I}) \rightarrow \text{End}(E(\mathbb{I}\mathbb{N}_\infty^n))$ is a homomorphism. $\square$

The following theorem describes the structure of the semigroup $\mathbb{I}\mathbb{N}_\infty^n$.

Theorem 4.6. *Let n be any positive integer ≥ 2 . Then the semigroup $\mathbb{I}\mathbb{N}_\infty^n$ is isomorphic to the semidirect product $\mathcal{S}_n \ltimes_{\mathfrak{h}} (\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ of free semilattice with the unit $(\mathcal{P}_\infty(\mathbb{N}^n), \cup)$ by the symmetric group $\mathcal{S}_n$.*

Proof. Since by Theorem 3.2 the group of units $H(\mathbb{I})$ of the semigroup $\mathbb{I}\mathbb{N}_\infty^n$ is isomorphic to the symmetric group $\mathcal{S}_n$, it is sufficient to show that the semigroup $\mathbb{I}\mathbb{N}_\infty^n$ is isomorphic to the semidirect product $H(\mathbb{I}) \ltimes_{\mathfrak{h}} E(\mathbb{I}\mathbb{N}_\infty^n)$ the semilattice $E(\mathbb{I}\mathbb{N}_\infty^n)$ by the group of units $H(\mathbb{I})$ of $\mathbb{I}\mathbb{N}_\infty^n$ with the respect to the homomorphism

$$\mathfrak{h}: H(\mathbb{I}) \rightarrow \text{End}(E(\mathbb{I}\mathbb{N}_\infty^n)), \quad \sigma \mapsto \mathfrak{h}_\sigma,$$

where $(\varepsilon)\mathfrak{h}_\sigma = \sigma^{-1}\varepsilon\sigma$, $\varepsilon \in E(\mathbf{IN}_\infty^n)$.

We define a map $\mathfrak{T}: \mathbf{IN}_\infty^n \rightarrow H(\mathbb{I}) \ltimes_{\mathfrak{h}} E(\mathbf{IN}_\infty^n)$ in the following way

$$(\alpha)\mathfrak{T} = (\alpha_\sigma, \alpha^{-1}\alpha),$$

where the element α_σ of the group of units $H(\mathbb{I})$ of $\mathbf{IN}_\infty^n$ is defined by formula $\alpha \mapsto \alpha_\sigma$ in Theorem 3.8. By Lemma 3.5 and Theorem 3.8 the map $\mathfrak{T}: \mathbf{IN}_\infty^n \rightarrow H(\mathbb{I}) \ltimes_{\mathfrak{h}} E(\mathbf{IN}_\infty^n)$ is well defined and by Theorem 3.9(iv) it is surjective. Suppose there exist $\alpha, \beta \in \mathbf{IN}_\infty^n$ such that $(\alpha)\mathfrak{T} = (\beta)\mathfrak{T}$. Then $(\alpha_\sigma, \alpha^{-1}\alpha) = (\beta_\sigma, \beta^{-1}\beta)$ and by Corollary 3.7 we get that

$$\alpha = \alpha_\sigma \alpha^{-1} \alpha = \beta_\sigma \beta^{-1} \beta = \beta,$$

and hence the map $\mathfrak{T}: \mathbf{IN}_\infty^n \rightarrow H(\mathbb{I}) \ltimes_{\mathfrak{h}} E(\mathbf{IN}_\infty^n)$ is injective.

Fix arbitrary $\alpha, \beta \in \mathbf{IN}_\infty^n$. Then Lemma 3.5 and Theorem 3.8 imply that $\alpha\beta\mathfrak{C}_{\text{mg}}\alpha_\sigma\beta_\sigma$, and since $\alpha_\sigma, \beta_\sigma \in H(\mathbb{I})$, we get that $\alpha_\sigma\beta_\sigma = (\alpha\beta)_\sigma$. This implies that

$$\begin{aligned} (\alpha)\mathfrak{T}(\beta)\mathfrak{T} &= (\alpha_\sigma, \alpha^{-1}\alpha) (\beta_\sigma, \beta^{-1}\beta) = \\ &= (\alpha_\sigma\beta_\sigma, \beta_\sigma^{-1}\alpha^{-1}\alpha\beta_\sigma\beta^{-1}\beta) = \\ &= ((\alpha\beta)_\sigma, \beta_\sigma^{-1}\alpha^{-1}\alpha\beta_\sigma\beta^{-1}\beta). \end{aligned}$$

By Lemma 4.5 the map $\mathfrak{h}_\sigma: E(\mathbf{IN}_\infty^n) \rightarrow E(\mathbf{IN}_\infty^n): \alpha \mapsto \sigma^{-1}\alpha\sigma$ is an automorphism of the semilattice $E(\mathbf{IN}_\infty^n)$, and hence we get that $\beta_\sigma^{-1}\alpha^{-1}\alpha\beta_\sigma$ is an idempotent of the semigroup $\mathbf{IN}_\infty^n$. By Lemma 3.5 and Corollary 3.7 for any $\alpha \in \mathbf{IN}_\infty^n$ there exists the unique element α_σ of the group of units $H(\mathbb{I})$ such that $\alpha \leq \alpha_\sigma$. Lemma 1.4.6 from [14] implies that $\beta = \beta_\sigma\beta^{-1}\beta$, and hence

$$\begin{aligned} \beta_\sigma^{-1}\alpha^{-1}\alpha\beta_\sigma\beta^{-1}\beta &= (\beta_\sigma^{-1}\alpha^{-1}\alpha\beta_\sigma) (\beta^{-1}\beta) (\beta^{-1}\beta) = \\ &= (\beta^{-1}\beta\beta_\sigma^{-1})(\alpha^{-1}\alpha)(\beta_\sigma\beta^{-1}\beta) = \\ &= (\beta_\sigma\beta^{-1}\beta)^{-1}(\alpha^{-1}\alpha)(\beta_\sigma\beta^{-1}\beta) = \\ &= \beta^{-1}(\alpha^{-1}\alpha)\beta = \\ &= (\beta^{-1}\alpha^{-1})(\alpha\beta) = \\ &= (\alpha\beta)^{-1}(\alpha\beta). \end{aligned}$$

Therefore, we have that

$$(\alpha\beta)\mathfrak{T} = (\sigma_{\alpha\beta}, (\alpha\beta)^{-1}\alpha\beta) = (\alpha)\mathfrak{T}(\beta)\mathfrak{T},$$

and hence the map $\mathfrak{T}: \mathbf{IN}_\infty^n \rightarrow H(\mathbb{I}) \ltimes_{\mathfrak{h}} E(\mathbf{IN}_\infty^n)$ is a homomorphism, which completes the proof of the theorem. $\square$

Remark 4.7. The monoid $\mathbf{IN}_\infty$ of all partial cofinite isometries of the set of positive integers $\mathbb{N}$ is a submonoid of the monoid $\mathcal{J}_\infty^{\nearrow}(\mathbb{N})$ of cofinite,

monotone, non-decreasing, injective partial transformations of $\mathbb{N}$, [11]. The structure of the semigroup is described in [12]. Also, Lemma 3.5 implies that for any positive integer $n \geq 2$ the semigroup $\mathbf{IN}_{\infty}^n$ is an inverse submonoid of the monoid $\mathcal{PO}_{\infty}(\mathbb{N}_{\leq}^n)$ of monotone injective partial selfmaps of $\mathbb{N}_{\leq}^n$ with the order product having cofinite domain and image, [6, 7, 8].

ACKNOWLEDGEMENTS

The authors acknowledge Alex Ravsky for his comments and suggestions. The authors gratefully acknowledge the referee for several comments and suggestions which have considerably improve the original version of the manuscript.

REFERENCES

- [1] O. Bezushchak. On growth of the inverse semigroup of partially defined co-finite automorphisms of integers. *Algebra Discrete Math.*, (2):45–55, 2004.
- [2] O. Bezushchak. Green’s relations of the inverse semigroup of partially defined co-finite isometries of discrete line. *Visn., Ser. Fiz.-Mat. Nauky, Kyiv. Univ. Im. Tarasa Shevchenka*, (1):12–16, 2008.
- [3] A. H. Clifford, G. B. Preston. *The algebraic theory of semigroups. Vol. I.* Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1961.
- [4] A. H. Clifford, G. B. Preston. *The algebraic theory of semigroups. Vol. II.* Mathematical Surveys, No. 7. American Mathematical Society, Providence, R.I., 1967.
- [5] J. A. Green. On the structure of semigroups. *Ann. of Math. (2)*, 54:163–172, 1951, doi: 10.2307/1969317.
- [6] O. Gutik, O. Krokhmalna. The monoid of monotone injective partial selfmaps of the poset $(\mathbb{N}^n, \leq)$ with cofinite domains and images. *Preprint*.
- [7] O. Gutik, I. Pozdniakova. On the monoid of monotone injective partial selfmaps of $\mathbb{N}_{\leq}^2$ with co-finite domains and images. *Visn. L’viv. Univ., Ser. Mekh.-Mat.*, 81:100–116, 2016.
- [8] O. Gutik, I. Pozdniakova. On the monoid of monotone injective partial selfmaps of $\mathbb{N}_{\leq}^2$ with co-finite domains and images, II. *Visn. L’viv. Univ., Ser. Mekh.-Mat.*, 82:109–127, 2016.
- [9] O. Gutik, A. Savchuk. On the semigroup $\mathbf{ID}_{\infty}$. *Visn. Lviv. Univ., Ser. Mekh.-Mat.*, 83:5–19, 2017.
- [10] O. Gutik, A. Savchuk. The semigroup of partial co-finite isometries of positive integers. *Bukovyn. Mat. Zh.*, 6(1–2):42–51, 2018.
- [11] O. Gutik, A. Savchuk. On inverse submonoids of the monoid of almost monotone injective co-finite partial selfmaps of positive integers. *Carpathian Mathematical Publications*, 11(2):arXiv:1904.11802, 2019.
- [12] Oleg Gutik, Dušan Repovš. Topological monoids of monotone injective partial selfmaps of $\mathbb{N}$ with cofinite domain and image. *Studia Sci. Math. Hungar.*, 48(3):342–353, 2011, doi: 10.1556/SScMath.48.2011.3.1176.
- [13] Oleg Gutik, Dušan Repovš. On monoids of injective partial cofinite selfmaps. *Math. Slovaca*, 65(5):981–992, 2015, doi: 10.1515/ms-2015-0067.
- [14] Mark V. Lawson. *Inverse semigroups*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998, doi: 10.1142/9789812816689. The theory of partial symmetries.

- [15] R. McFadden, L. O'Carroll. F -inverse semigroups. *Proc. London Math. Soc.* (3), 22:652–666, 1971, doi: [10.1112/plms/s3-22.4.652](https://doi.org/10.1112/plms/s3-22.4.652).
- [16] Mario Petrich. *Inverse semigroups*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. A Wiley-Interscience Publication.
- [17] Tôru Saitô. Proper ordered inverse semigroups. *Pacific J. Math.*, 15:649–666, 1965.
- [18] V. V. Wagner. Generalized groups. *Dokl. Akad. Nauk SSSR*, 84:1119–1122, 1952.

Oleg Gutik

FACULTY OF MATHEMATICS, IVAN FRANKO NATIONAL UNIVERSITY OF LVIV, UNIVERSITY
SYTETSKA 1, LVIV, 79000, UKRAINE

Email: oleg.gutik@lnu.edu.ua

ORCID: orcid.org/0000-0001-8513-0282

Anatolii Savchuk

FACULTY OF MATHEMATICS, IVAN FRANKO NATIONAL UNIVERSITY OF LVIV, UNIVERSITY
SYTETSKA 1, LVIV, 79000, UKRAINE

Email: asavchuk3333@gmail.com