Connection problems for the generalized hypergeometric Appell polynomials

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Abstract. Using a straightforward approach, we derived the solution of the inverse problem for the generalized hypergeometric Appell polynomials. Also, we established the recurrence formulas for the solutions of the connection problem between them and the Bernoulli polynomials, as well as between them and the Gould-Hopper polynomials and between two different generalized hypergeometric Appell polynomial families. In addition, we present one new recurrence identity for the generalized hypergeometric Appell polynomials.

Анотація. В статті використано загальний підхід до розв'язування задач зв'язності для многочленів Аппеля, який базується на тому, що відношення трансферних функцій, які представляють собою формальні степеневі ряди, даних двох сімейств многочленів Аппеля є відомим рядом. Використовуючи рекурентні формули для знаходження коефіцієнтів ряду, який є відношенням двох даних формальних степеневих рядів, ми отримали розв'язок оберненої задачі для узагальнених гіпергеометричних многочленів Аппеля. В загальному випадку розв'язок визначається рекурентними формулами, але у деяких часткових випадках, коли породжуюча функція має простий вигляд, розв'язок оберненої задачі виражається у замкнутій формі, зокрема, для многочленів Гюлда-Хоппера, або для узагальнених гіпергеометричних многочленів Аппеля, породжуюча функція яких співпадає із функцією Бесселя першого роду.

Користуючись цим же методом і відомим представленням узагальнених гіпергеометричних многочленів Аппеля у формі звичайного диференціального оператора, ми знайшли рекурентні формули розв'язку задачі зв'язності між узагальненими гіпергеометричними многочленами Аппеля та многочленами Бернуллі, між узагальненими гіпергеометричними многочленами Аппеля – многочленами Гюлд-Хоппера та між

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Ключові слова: многочлени Аппеля, задачі зв'язності, представлення груп, формальний степеневий ряд, рекурентні рівняння, диференціальні рівняння

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двома різними сімействами узагальнених гіпергеометричних многочленів Аппеля. Використовуючи схожий підхід, ми отримали нове рекурентне рівняння для узагальнених гіпергеометричних многочленів Аппеля, коефіцієнти якого визначаються рекурентно, і встановили замкнуту форму декількох перших з них. Частковими випадками отриманого рівняння є, зокрема, відомі рекурентні рівняння для многочленів Гоулда-Хоппера і для многочленів Ерміта.

Крім того, розв'язок задачі зв'язності для двох різних сімейств узагальнених гіпергеометричних многочленів Аппеля отримано в іншій формі – з використанням значень цих многочленів в нулі.

1. INTRODUCTION

Given two polynomial sequences $\{P_n(x)\}_{n \geq 0}$ and $\{Q_n(x)\}_{n \geq 0}$ such that $\deg P_n(x) = \deg Q_n(x) = n$, the connection problem between them is given by the expression

$$P_n(x) = \sum_{i=0}^n c_i(n) Q_i(x),$$

where the unknown coefficients $c_i(n)$ are need to be found. In the case, when $P_n(x) = x^n$, the connection problem is called the inverse problem

$$x^n = \sum_{i=0}^n c_i(n) Q_i(x).$$

Considering the classical polynomials involving the Laguerre, the Hermite, the ultraspherical, the Chebyshev of both kinds and other polynomials, the connection problem arose firstly as one of the steps leading to establishing of the polynomial sequences orthogonality. While the solutions of the inverse problem for the majority of the classical orthogonal polynomial sequences are known (one can be referred for examples to the fundamental manuscripts on the subject [1, 16, 14, 11]), the solution of the connection problems were found only for several polynomial sequences, such as the Hermite, the Laguerre and the Legendre ones [16].

Recently, applying different methods and techniques, researchers presented a large variety of the connection problems solutions for the orthogonal polynomial sequences. In [12], based on integral calculus, the Bernoulli-Laguerre and Euler-Laguerre connection problems solutions were obtained. Some authors derive the explicit solutions combining the hypergeometric representation and the inversion formulas, as it was made for the Hermite and other polynomials in [13, 8], when the others single out the connecting coefficients by virtue of algorithms based on the known recursive relations (see [9]), consequently, the obtained results frequently do not have the explicit form (see Meixner-Meixner connection problem in [17]).

Besides, usage of the computer algebra systems allows to derive the solutions which are rather cumbersome ones and are often expressed via the partial cases of the generalized hypergeometric function, to illustrate it we give here the Charlier-Meixner connection problem solution ([13]):

$$Q_n(x; \alpha, \beta, n) = \sum_{k=0}^n \binom{n}{k} \frac{(n + \alpha + \beta + 1)^{(k)}}{(1 - N)^{(k)} (\beta + 1)^{(k)}} \times \\ \times \left(\frac{\mu}{\mu - 1} \right)^k {}_3F_2 \left[\begin{matrix} k + \gamma, k - n, k + n + \alpha + \beta + 1 \\ k + \beta + 1, k + 1 - N \end{matrix} \middle| \frac{\mu}{\mu - 1} \right] m_n^{(\gamma, \mu)}(x),$$

where $Q_n(x; \alpha, \beta, n)$ and $m_n^{(\gamma, \mu)}(x)$ are the Charlier and the Meixner polynomials, respectively, and the generalized hypergeometric function ${}_pF_q$ is defined as follows

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right] = \sum_{i=0}^{\infty} \frac{a_1^{(i)} a_2^{(i)} \dots a_p^{(i)}}{b_1^{(i)} b_2^{(i)} \dots b_q^{(i)}} \frac{z^i}{i!},$$

where $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q$ are complex parameters and none of b_i is equal to a non-positive integer or zero, $x^{(n)}$ denotes the Pochhammer symbol (or rising factorial) defined by $x^{(n)} = x(x+1)(x+2) \dots (x+n-1)$ for $n \geq 1$ and $x^{(0)} = 1$.

Apart from the classical polynomial families, there exists a wide class of Appell polynomials $P_n(x)$ introduced in [3] which are characterized by the recurrence equation

$$P'_n(x) = nP_{n-1}(x). \quad (1.1)$$

The generating function $G(x, t)$ of an arbitrary Appell polynomials $P_n(x)$ satisfy the following condition

$$G(x, t) = \exp(xt)A(t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!},$$

where $A(t)$ is a formal power series in the form

$$A(t) = a_0 + a_1 \frac{t}{1!} + a_2 \frac{t^2}{2!} + \dots + a_n \frac{t^n}{n!} + \dots, \quad a_0 \neq 0,$$

where $a_i \in \mathbb{R}$ for all $i = 0, 1, \dots$

Appell polynomials include the monomials, the Euler, the Bernoulli, the Hermite, the Generalized Bernoulli, the Hypergeometric Bernoulli polynomials and others. The Appell polynomials are widely used in various fields of mathematics, in particular in the group representation theory, for example, see [4]. Some solutions of the connection problems between the single representatives of those class are known, for instance, the explicit form of the Bernoulli-Gould-Hopper connection problem ([18]).

In [5], the new Appell polynomial family $A_n^{(k)}(m, x)$ was presented. The generalized hypergeometric Appell polynomials $A_n^{(k)}(m, x)$ are expressed in the terms of the generalized hypergeometric function as follows

$$A_n^{(k)}(m, x) = x^n \cdot {}_{p+k}F_q \left[\begin{matrix} \Delta(k, -n), a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{m}{x^k} \right],$$

where

$$\Delta(k, -n) = -\frac{n}{k}, -\frac{n-1}{k}, \dots, -\frac{n-k+1}{k}.$$

In the case when $\Delta(k, -n)$ is the empty set, $k = m$, and $m = (-1)^k h k^k$ the generalized hypergeometric Appell polynomials turns into the Gould-Hopper polynomials $g_n^m(x, h)$ ([10]).

The aim of this paper is to find the solution of the inverse problem for the generalized hypergeometric Appell polynomials as well as to solve the connection problems between them and the Bernoulli polynomials, between two different families of the generalized hypergeometric Appell polynomials, and, finally, between the generalized hypergeometric Appell polynomials and the Gould-Hopper polynomials in order to show both families do differ each other.

We use the general method proposed in [6]. The fact is that the solution of the connection problem for two given Appell polynomial families $\{P_n(x)\}_{n \geq 0}$ and $\{Q_n(x)\}_{n \geq 0}$ with generating functions $G_1(x, t)$ and $G_2(x, t)$, respectively,

$$G_1(x, t) = \exp(xt)A_1(t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!},$$

$$G_2(x, t) = \exp(xt)A_2(t) = \sum_{n=0}^{\infty} Q_n(x) \frac{t^n}{n!},$$

where by the transfer functions of $P_n(x)$ and $Q_n(x)$ we mean the parts $A_1(t)$ and $A_2(t)$ of generating functions $G_1(x, t)$ and $G_2(x, t)$, respectively, has the following form

$$Q_n(x) = \sum_{i=0}^n \frac{n!}{i!} \alpha_{n-i} P_i(x), \quad \text{where} \quad \frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty} \alpha_k t^k. \quad (1.2)$$

In what follow, we denote for brevity

$$(a)_{p,i} = a_1^{(i)} a_2^{(i)} \dots a_p^{(i)}, \quad (b)_{q,i} = b_1^{(i)} b_2^{(i)} \dots b_q^{(i)}.$$

Then the generalized hypergeometric function is written in the following manner

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right] = \sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{z^i}{i!}.$$

2. MAIN RESULTS ON THE GENERALIZED HYPERGEOMETRIC APPELL POLYNOMIALS

Let us start from the one property of the generalized hypergeometric function.

Lemma 2.1. *An exponential formal series $A(t) = \sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{t^i}{i!}$ is invertible if and only if $\frac{a_{p,0}}{b_{q,0}} \neq 0$, and its inverse $1/A(t)$ can be computed as follows*

$$\frac{1}{A(t)} = \sum_{n=0}^{\infty} c_n t^n,$$

where $c_0 = \frac{a_{p,0}}{b_{q,0}}$ and the coefficients c_n satisfy the recursive relation

$$c_n = - \sum_{i=1}^n \frac{(a)_{p,i}}{(b)_{q,i}} \frac{c_{n-i}}{i!}.$$

Proof. We are looking for the power series $\sum_{n=0}^{\infty} c_n t^n$ such that

$$A(t) \cdot \left(\sum_{n=0}^{\infty} c_n t^n \right) = 1.$$

Expanding the latter equality

$$\left(1 + \frac{(a)_{p,1}}{(b)_{q,1}} \frac{t}{1!} + \frac{(a)_{p,2}}{(b)_{q,2}} \frac{t^2}{2!} + \frac{(a)_{p,3}}{(b)_{q,3}} \frac{t^3}{3!} + \dots \right) \times \\ \times (c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \dots) = 1,$$

and equating the equal powers of t on the both sides, we obtain the system of linear equations

$$\begin{aligned} t^0 : \quad & c_0 = 1, \\ t^1 : \quad & c_1 + \frac{(a)_{p,1}}{(b)_{q,1}} c_0 = 0, \\ t^2 : \quad & c_2 + \frac{(a)_{p,1}}{(b)_{q,1}} c_1 + \frac{(a)_{p,2}}{(b)_{q,2}} \frac{1}{2!} c_0 = 0, \\ t^3 : \quad & c_3 + \frac{(a)_{p,1}}{(b)_{q,1}} c_2 + \frac{(a)_{p,2}}{(b)_{q,2}} \frac{1}{2!} c_1 + \frac{(a)_{p,3}}{(b)_{q,3}} \frac{1}{3!} c_0 = 0, \\ \dots \\ t^n : \quad & c_n + \sum_{i=1}^n \frac{(a)_{p,i}}{(b)_{q,i}} \frac{c_{n-i}}{i!} = 0, \end{aligned}$$

and so on, whence c_i is defined in a recursive way

$$\begin{aligned}
 c_0 &= 1, \\
 c_1 &= -\frac{(a)_{p,1}}{(b)_{q,1}}, \\
 c_2 &= \frac{((a)_{p,1})^2}{((b)_{q,1})^2} - \frac{1}{2} \frac{(a)_{p,2}}{(b)_{q,2}}, \\
 c_3 &= -\frac{((a)_{p,1})^3}{((b)_{q,1})^3} + \frac{(a)_{p,1}(a)_{p,2}}{(b)_{q,1}(b)_{q,2}} - \frac{1}{6} \frac{(a)_{p,3}}{(b)_{q,3}}, \\
 &\dots \\
 c_n &= -\sum_{i=1}^n \frac{(a)_{p,i}}{(b)_{q,i}} \frac{c_{n-i}}{i!}, \\
 &\dots
 \end{aligned}
 \quad \square$$

Now we can obtain the solution of the inversion problems to the generalized hypergeometric Appell polynomials as well as their explicit form.

Theorem 2.2. (i) *The generalized hypergeometric Appell polynomials has the representation*

$$A_n^{(k)}(m, x) = n! \sum_{i=0}^{\left[\frac{n}{k}\right]} \frac{(-1)^{ki} (a)_{p,i} m^i}{i! (n-ki)! k^{ki} (b)_{q,i}} x^{n-ki}.$$

(ii) *The solution of the inverse problem of the generalized hypergeometric Appell polynomials has the form*

$$x^n = n! \sum_{i=0}^{\left[\frac{n}{k}\right]} \frac{c_i}{(n-ki)!} A_{n-ki}^{(k)}(m, x),$$

where $c_0 = 1$, and

$$c_i = -\sum_{j=1}^{\left[\frac{i}{k}\right]} \frac{(-1)^{kj} (a)_{p,j} m^j}{j! k^{kj} (b)_{q,j}} c_{i-j}.$$

Proof. For the proof of (i) see [4].

(ii) Vice versa, the ratio of the transfer functions of the monomials the generalized hypergeometric Appell polynomials $A_n^{(k)}(m, x)$ has the form

$$\frac{A_2(t)}{A_1(t)} = \frac{1}{A_1(t)} = \frac{1}{{}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| (-1)^k \frac{mt^k}{k^k} \right]} =$$

$$= \frac{1}{\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{(-1)^{ki} m^i}{k^{ki}} \frac{t^{ki}}{i!}}.$$

Thus, we are searching for the reciprocal of the generalized hypergeometric Appell polynomials $A_n^{(k)}(m, x)$ in the form of the series $\sum_{i=0}^{\infty} c_i t^{ki}$ with unknown coefficients c_i such that

$$\frac{1}{A_1(t)} = \sum_{i=0}^{\infty} c_i t^{ki}.$$

The application of lemma 2.1 allows us to derive those coefficients as follows

$$c_0 = 1, \quad c_i = - \sum_{j=1}^{\left[\frac{i}{k}\right]} \frac{(a)_{p,j}}{(b)_{q,j}} \frac{(-1)^{kj} m^j}{k^{kj}} \frac{c_{i-j}}{j!},$$

where the first of them are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= - \left(\frac{(a)_{p,1}}{(b)_{q,1}} \right) \frac{m}{(-k)^k}, \\ c_2 &= \left(\frac{((a)_{p,1})^2}{((b)_{q,1})^2} - \frac{1}{2} \frac{(a)_{p,2}}{(b)_{q,2}} \right) \left(\frac{m}{(-k)^k} \right)^2, \\ c_3 &= \left(- \frac{((a)_{p,1})^3}{((b)_{q,1})^3} + \frac{(a)_{p,1}(a)_{p,2}}{(b)_{q,1}(b)_{q,2}} - \frac{1}{6} \frac{(a)_{p,3}}{(b)_{q,3}} \right) \left(\frac{m}{(-k)^k} \right)^3, \end{aligned}$$

and, with the formula (1.2) of the connection problem, we obtain (ii). $\square$

In the case when the transfer power series are expressed by the confluent generalized hypergeometric function, the solution of the inverse problem of the generalized hypergeometric Appell polynomials is simplified significantly.

Example 2.3. In the case of the Gould-Hopper polynomials $g_n^m(x, h)$, their transfer function $A_{gh}(t)$ simplified to the exponential one

$$\begin{aligned} A_{gh}(t) &= {}_0F_0 \left[- \middle| ht^m \right] = \exp(ht^m) = \\ &= 1 + ht^m \frac{1}{1!} + (ht^m)^2 \frac{1}{2!} + (ht^m)^3 \frac{1}{3!} + \cdots, \end{aligned}$$

with the simple reciprocal series

$$\frac{1}{A_{gh}(t)} = A_{gh}(t)^{-1} = \exp(-ht^m) = 1 - ht^m \frac{1}{1!} + (ht^m)^2 \frac{1}{2!} - (ht^m)^3 \frac{1}{3!} + \dots,$$

here from, using the connection formula, we immediately obtain the solution of the inverse problem of the Gould-Hopper polynomials

$$x^n = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{n!}{k!} \frac{(-h)^k}{(n - km)!} g_{n-km}^m(x, h),$$

which coincides with results [10].

Example 2.4. The partial case of the confluent generalized hypergeometric function is expressed via Bessel function as follows

$${}_0F_1 \left[\begin{matrix} - \\ - \end{matrix} \middle| z \right] = J_0(2\sqrt{z}),$$

its reciprocal series is known (see [7])

$$\frac{1}{J_0(2\sqrt{z})} = \sum_{m=0}^{\infty} \omega_m \frac{z^m}{(m!)^2},$$

coefficients ω_m of which form the sequence number A000275 at the on-line encyclopedia of integer sequences OEIS ([2]).

Thus, having the transfer function

$$A_{bs}(t) = {}_0F_1 \left[\begin{matrix} - \\ 1 \end{matrix} \middle| (-1)^k \frac{mt^k}{k^k} \right] = J_0 \left(2\sqrt{\frac{mt^k}{(-k)^k}} \right)$$

with the reciprocal one

$$\frac{1}{A_{bs}(t)} = \sum_{i=0}^{\infty} \omega_i \frac{t^{ki}}{(i!)^2} \left(\frac{m}{(-k)^k} \right)^i,$$

we obtain the explicit formula of the inverse problem solution

$$x^n = \sum_{i=0}^{\left[\frac{n}{k}\right]} \frac{n!}{(n - ki)!} \frac{\omega_i}{(i!)^2} \left(\frac{m}{(-k)^k} \right)^i A_{n-ki}^{(k)}(m, x)$$

for the polynomials $A_n^{(k)}(m, x) = x^n {}_kF_1 \left[\begin{matrix} \Delta(k, -n) \\ 1 \end{matrix} \middle| \frac{m}{x^k} \right]$.

Considering the Bernoulli polynomials $B_n(x)$ which possess the following transfer function

$$A_{br}(t) = \frac{t}{\exp(t) - 1}$$

and applying the same method, we can construct the solution of the connection problem for the Bernoulli polynomial families and the generalized hypergeometric Appell ones.

Theorem 2.5. *The solution of the connection problem between the generalized hypergeometric Appell polynomials and the Bernoulli ones has the form*

$$(i) \quad A_n^{(k)}(m, x) = \sum_{i=0}^n \frac{n!}{i!} \left(\sum_{j=0}^{\left[\frac{i}{k}\right]} \frac{m^j (-1)^{kj} (a)_{p,j}}{j! k^{kj} (n-i-kj+1)! (b)_{q,j}} \right) B_i(x),$$

$$(ii) \quad B_n(x) = \sum_{i=0}^n \frac{n!}{i!} \left(\sum_{j=0}^{\left[\frac{n-i}{k}\right]} \frac{m^j (-1)^{kj} (a)_{p,j}}{j! k^{kj} (n-i-kj)! (b)_{q,j}} B_{n-i-kj} \right) A_i^{(k)}(m, x).$$

Proof. (i) Expressing the ratio of the corresponding transfer functions, we have

$$\frac{A_1(t)}{A_{br}(t)} = \frac{\left(\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{(-1)^{ki} m^i t^{ki}}{k^{ki} i!} \right) (\exp(t) - 1)}{t}$$

We are searching for the series $\sum_{i=0}^{\infty} c_i t^i$ such that

$$\left(\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{(-1)^{ki} m^i t^{ki}}{k^{ki} i!} \right) (\exp(t) - 1) = t \left(\sum_{i=0}^{\infty} c_i t^i \right).$$

After simplification we get

$$\left(\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{(-1)^{ki} m^i t^{ki}}{k^{ki} i!} \right) (\exp(t) - 1) = \sum_{r=0}^{\infty} \left(\sum_{j=0}^{\left[\frac{r}{k}\right]} \frac{m^j (-1)^{kj} (a)_{p,j}}{j! k^{kj} (r-kj+1)! (b)_{q,j}} \right) t^r,$$

which with the use of (1.2) immediately gives the (i) result.

(ii) Vice versa, from

$$\frac{A_{br}(t)}{A_1(t)} = \frac{t}{\left(\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{(-1)^{ki} m^i t^{ki}}{k^{ki} i!} \right) (\exp(t) - 1)} = \sum_{i=0}^{\infty} d_i t^i$$

we deduce that

$$t = \left(\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{(-1)^{ki} m^i t^{ki}}{k^{ki} i!} \right) (\exp(t) - 1) = \sum_{i=0}^{\infty} d_i t^i,$$

or

$$t = \left(\sum_{r=0}^{\infty} \left(\sum_{j=0}^{\left\lfloor \frac{r}{k} \right\rfloor} \frac{m^j (-1)^{kj} (a)_{p,j}}{j! k^{kj} (n-r-kj+1)! (b)_{q,j}} \right) t^{r+1} \right) = \sum_{i=0}^{\infty} d_i t^i,$$

wherefrom

$$d_j = \sum_{j=0}^{\left\lfloor \frac{n-r}{k} \right\rfloor} \frac{m^j (-1)^{kj} (a)_{p,j}}{j! k^{kj} (n-r-kj)! (b)_{q,j}} B_{n-r-kj},$$

where B_n denote the Bernoulli number, and applying the formula (1.2) for the connection problem ends the proof. $\square$

Now we are going to establish the connection between the generalized hypergeometric Appell polynomials and the Gould-Hopper polynomials and the connection between the two different generalized hypergeometric Appell polynomial families.

Expressing the ratio of the transfer functions the generalized hypergeometric Appell polynomials and the Gould-Hopper polynomials, we have

$$\frac{A_1(t)}{A_{gh}(t)} = \frac{\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{m^i}{(-k)^{ki}} \frac{t^{ki}}{i!}}{\exp(mt^k)}$$

We are searching for the series $\sum_{i=0}^{\infty} f_i t^i$ such that

$$\sum_{i=0}^{\infty} \frac{(a)_{p,i}}{(b)_{q,i}} \frac{m^i}{(-k)^{ki}} \frac{t^{ki}}{i!} = \exp(mt^k) \left(\sum_{i=0}^{\infty} f_i t^i \right).$$

Equating the equal powers of t , we obtain the system of linear equations

$$\begin{aligned} f_0 &= 1, & f_1 &= \cdots = f_{k-1} = 0, \\ f_k + m f_0 &= \frac{m}{(-k)^k} \frac{(a)_{p,1}}{(b)_{q,1}} \frac{1}{1!}, & f_{k+1} &= \cdots = f_{2k-1} = 0, \\ f_{2k} + \frac{m}{1!} f_k + \frac{m^2}{2!} f_0 &= \left(\frac{m}{(-k)^k} \right)^2 \frac{(a)_{p,2}}{(b)_{q,2}} \frac{1}{2!}, & f_{2k+1} &= \cdots = f_{3k-1} = 0, \\ f_{3k} + \frac{m}{1!} f_{2k} + \frac{m^2}{2!} f_k + \frac{m^3}{3!} f_0 &= \\ &= \left(\frac{m}{(-k)^k} \right)^3 \frac{(a)_{p,3}}{(b)_{q,3}} \frac{1}{3!}, & f_{3k+1} &= \cdots = f_{4k-1} = 0, \\ &\dots\dots\dots \\ f_{ki} + \frac{m}{1!} f_{k(i-1)} + \cdots + \frac{m^i}{i!} f_0 &= \\ &= \left(\frac{m}{(-k)^k} \right)^i \frac{(a)_{p,i}}{(b)_{q,i}} \frac{1}{i!}, & f_{ki+1} &= \cdots = f_{k(i+1)-1} = 0, \\ &\dots\dots\dots \end{aligned}$$

wherefrom

$$f_0 = 1, \quad f_{ki} = \left(\frac{m}{(-k)^k} \right)^i \frac{(a)_{p,i}}{(b)_{q,i}} \frac{1}{i!} - \sum_{j=0}^{i-1} \frac{m^{j+1}}{(j+1)!} c_{k(i-j-1)},$$

and applying the formula (1.2) of the connection problem we obtain the solution of the generalized hypergeometric Appell - the Gould-Hopper connection problem.

Proceeding the same scheme with the inverse transfer functions ratio, we obtain the solution of the Gould-Hopper - the generalized hypergeometric Appell connection problem.

Besides, given two different generalized hypergeometric Appell polynomial families, $A_n^{(k)}(m, x)$ and $A_n^{(k_1)}(m_1, x)$ with transfer functions $A_1(t)$ and $A_3(t)$, respectively,

$$A_1(t) = {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{mt^k}{(-k)^k} \right], \quad A_3(t) = {}_\tau F_\rho \left[\begin{matrix} \alpha_1, \dots, \alpha_\tau \\ \beta_1, \dots, \beta_\rho \end{matrix} \middle| \frac{m_1 t^{k_1}}{(-k_1)^{k_1}} \right].$$

we obtain the solution of the connection problem of two different generalized hypergeometric Appell polynomial families as follows

$$A_n^{(k)}(m, x) = \sum_{i=0}^n \frac{n!}{(n-i)!} \gamma_i A_{n-i}^{(k_1)}(m_1, x),$$

where $\gamma_0 = 1$ and coefficients γ_i satisfy the recurrence equations

$$\begin{cases} \sum_{j=0}^{\left\lfloor \frac{n}{k_1} \right\rfloor} \gamma_{n-k_1 j} \left(\frac{m_1}{(-k_1)^{k_1}} \right)^j \frac{(\alpha)_{p,j}}{(\beta)_{q,j}} \frac{1}{j!} = \left(\frac{m}{(-k)^k} \right)^{\frac{n}{k}} \frac{(a)_{p, \frac{n}{k}}}{(b)_{q, \frac{n}{k}}} \frac{1}{\left(\frac{n}{k} \right)!}, & \text{if } n \neq 0 \bmod k, \\ \sum_{j=0}^{\left\lfloor \frac{n}{k_1} \right\rfloor} \gamma_{n-k_1 j} \left(\frac{m_1}{(-k_1)^{k_1}} \right)^j \frac{(\alpha)_{p,j}}{(\beta)_{q,j}} \frac{1}{j!} = 0, & \text{if } n = 0 \bmod k \text{ or } n < k_1. \end{cases}$$

Thus, we proved the following statement.

Theorem 2.6. *The solution of the connection problem between the generalized hypergeometric Appell polynomials and the Gould-Hopper ones has the form*

(i) $A_n^{(k)}(m, x) = \sum_{i=0}^{\left[\frac{n}{k}\right]} \frac{n!}{(n-ki)!} f_{ki}(x) g_{n-ki}^{(k)}(x, m)$, where coefficients f_{ki} satisfy the recurrence equation

$$f_0 = 1, \quad f_{ki} = \left(\frac{m}{(-k)^k} \right)^i \frac{(a)_{p,i}}{(b)_{q,i}} \frac{1}{i!} - \sum_{j=0}^{i-1} \frac{m^{j+1}}{(j+1)!} f_{k(i-j-1)}.$$

(ii) $g_{ki}^{(k)}(x, m) = \sum_{i=0}^{\left[\frac{n}{k}\right]} \frac{n!}{(n-ki)!} l_{ki}(x) A_{n-ki}^{(k)}(m, x)$, where coefficients l_{ki} satisfy the recurrence equations

$$l_0 = 1, \quad l_{ki} = \frac{m^i}{i!} - \sum_{j=0}^{i-1} \left(\frac{m}{(-k)^k} \right)^{j+1} \frac{(a)_{p,j+1}}{(b)_{q,j+1}} \frac{1}{(j+1)!} l_{k(i-j-1)}.$$

(iii) The solution of the connection problem between two different generalized hypergeometric Appell polynomial families

$$A_n^{(k)}(m, x) = x^n {}_{p+k}F_q \left[\begin{matrix} \Delta(k, -n), a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| \frac{m}{x^k} \right],$$

$$A_n^{(k_1)}(m_1, x) = x^n {}_{\tau+k_1}F_\rho \left[\begin{matrix} \Delta(k_1, -n), \alpha_1, \dots, \alpha_\tau \\ \beta_1, \dots, \beta_\rho \end{matrix} \middle| \frac{m_1}{x^{k_1}} \right]$$

has the form

$$A_n^{(k)}(m, x) = \sum_{i=0}^n \frac{n!}{(n-i)!} \gamma_i A_{n-i}^{(k_1)}(m_1, x),$$

where $\gamma_0 = 1$ and coefficients γ_i satisfy the following recurrence equations

$$\left\{ \begin{array}{l} \sum_{j=0}^{\left[\frac{n}{k_1}\right]} \gamma_{n-k_1j} \left(\frac{m_1}{(-k_1)^{k_1}} \right)^j \frac{(\alpha)_{p,j}}{(\beta)_{q,j}} \frac{1}{j!} = \left(\frac{m}{(-k)^k} \right)^{\frac{n}{k}} \frac{(a)_{p,\frac{n}{k}}}{(b)_{q,\frac{n}{k}}} \frac{1}{\left(\frac{n}{k}\right)!}, \\ \sum_{j=0}^{\left[\frac{n}{k_1}\right]} \gamma_{n-k_1j} \left(\frac{m_1}{(-k_1)^{k_1}} \right)^j \frac{(\alpha)_{p,j}}{(\beta)_{q,j}} \frac{1}{j!} = 0, \end{array} \right. \quad \begin{array}{l} \text{if } n \neq 0 \bmod k, \\ \text{if } n = 0 \bmod k \text{ or } n < k_1. \end{array}$$

In addition, using similar considerations, we obtain new identity.

Theorem 2.7. Each generalized hypergeometric Appell polynomial $A_n^{(k)}(m, x)$ satisfies the following $(k\left[\frac{n}{k}\right]+2)$ -terms recurrent equation

$$A_n^{(k)}(m, x) - \alpha_{n-1} x A_{n-1}^k(m, x) - \alpha_{n-k} A_{n-k}^k(m, x) -$$

$$- \alpha_{n-2k} A_{n-2k}^k(m, x) - \cdots - \alpha_{n-\lceil \frac{n}{k} \rceil k} A_{n-\lceil \frac{n}{k} \rceil k}^k(m, x) = 0,$$

where α_{n-1} is defined via the system of recurrent equations

$$\begin{aligned} \alpha_{n-1} &= 1, \\ \alpha_{n-k} &= \beta_{n-k,n} - \alpha_{n-1} \beta_{n-1-k,n-1}, \\ \alpha_{n-2k} &= \beta_{n-2k,n} - \alpha_{n-1} \beta_{n-1-2k,n-1} - \alpha_{n-k} \beta_{n-2k,n-k}, \\ \alpha_{n-3k} &= \beta_{n-3k,n} - \alpha_{n-1} \beta_{n-1-3k,n-1} - \\ &\quad - \alpha_{n-k} \beta_{n-3k,n-k} - \alpha_{n-2k} \beta_{n-3k,n-2k}, \\ &\dots \\ \alpha_{n-\lceil \frac{n}{k} \rceil k} &= \beta_{n-\lceil \frac{n}{k} \rceil k,n} - \alpha_{n-1} \beta_{n-1-\lceil \frac{n}{k} \rceil k,n-1} - \alpha_{n-k} \beta_{n-\lceil \frac{n}{k} \rceil k,n-k} - \\ &\quad - \alpha_{n-2k} \beta_{n-\lceil \frac{n}{k} \rceil k,n-2k} - \cdots - \alpha_{n-(\lceil \frac{n}{k} \rceil - 1)k} \beta_{n-\lceil \frac{n}{k} \rceil k,n-(\lceil \frac{n}{k} \rceil - 1)k}, \end{aligned}$$

and

$$\beta_{n-kr,n} = \frac{(-1)^{kr} m^r(a)_{p,r}}{k^{kr} (b)_{q,r} r!} n(n-1) \cdots (n-kr+1).$$

Proof. Put $\beta_{n-kr,n} = \frac{(-1)^{kr} m^r(a)_{p,r}}{k^{kr} (b)_{q,r} r!} n(n-1) \cdots (n-kr+1)$. Then, according to the differential operator formula (see [5]), an arbitrary generalized hypergeometric Appell polynomials $A_n^{(k)}(m, x)$ could be written in our notations as follows

$$\begin{aligned} A_n^{(k)}(m, x) &= \left(\sum_{r=0}^{\lceil \frac{n}{k} \rceil} \frac{(-1)^{kr} m^r(a)_{p,r}}{k^{kr} (b)_{q,r} r!} D^{kr} \right) x^n = \\ &= x^n + \sum_{r=1}^{\lceil \frac{n}{k} \rceil} \frac{(-1)^{kr} m^r(a)_{p,r}}{k^{kr} (b)_{q,r} r!} n(n-1) \cdots (n-kr+1) x^{n-kr} = \\ &= x^n + \sum_{r=1}^{\lceil \frac{n}{k} \rceil} \beta_{n-kr,n} x^{n-kr}. \end{aligned}$$

Substituting the corresponding expressions for $A_n^{(k)}(m, x)$, $A_{n-1}^{(k)}(m, x)$, and $A_{n-k}^{(k)}(m, x)$ into the theorem statement formula and equating the equal powers of x , we obtain the following system

$$\begin{aligned} x^n : \quad & 1 - \alpha_{n-1} = 0, \\ x^{n-k} : \quad & \beta_{n-k,n} - \alpha_{n-1} \beta_{n-1-k,n-1} - \alpha_{n-k} = 0, \\ x^{n-2k} : \quad & \beta_{n-2k,n} - \alpha_{n-1} \beta_{n-1-2k,n-1} - \alpha_{n-k} \beta_{n-2k,n-k} - \alpha_{n-2k} = 0, \\ x^{n-3k} : \quad & \beta_{n-3k,n} - \alpha_{n-1} \beta_{n-1-3k,n-1} - \alpha_{n-k} \beta_{n-3k,n-k} - \end{aligned}$$

$$- \alpha_{n-2k} \beta_{n-3k, n-2k} - \alpha_{n-3k} = 0,$$

...

$$\begin{aligned} x^{n-\lfloor \frac{n}{k} \rfloor k} : \quad & \beta_{n-\lfloor \frac{n}{k} \rfloor k, n} - \alpha_{n-1} \beta_{n-1-\lfloor \frac{n}{k} \rfloor k, n-1} - \\ & - \alpha_{n-k} \beta_{n-\lfloor \frac{n}{k} \rfloor k, n-k} - \alpha_{n-2k} \beta_{n-\lfloor \frac{n}{k} \rfloor k, n-2k} - \dots - \\ & - \alpha_{n-(\lfloor \frac{n}{k} \rfloor - 1)k} \beta_{n-\lfloor \frac{n}{k} \rfloor k, n-(\lfloor \frac{n}{k} \rfloor - 1)k} - \alpha_{n-\lfloor \frac{n}{k} \rfloor k} = 0. \end{aligned}$$

From the first equation, we have $\alpha_{n-1} = 1$. Substituting adherently the previous expression for α_i of into the next equation, we obtain some first coefficients

$$\begin{aligned} \alpha_{n-k} &= \frac{(-1)^k m(a)_{p,1} k(n-1) \cdots (n-k+1)}{k^k (b)_{q,1}} - \\ & - \frac{(-1)^k m(a)_{p,1} k(n-1)(n-2) \cdots (n-k+1)}{k^k (b)_{q,1}} = \\ & = \frac{(-1)^k k m(a)_{p,1} (n-1) \cdots (n-k+1)}{k^k (b)_{q,1}}, \\ \alpha_{n-2k} &= \frac{(-1)^{2k} m^2(a)_{p,2} 2k(n-1) \cdots (n-2k+1)}{k^{2k} (b)_{q,2} 2!} - \\ & - \frac{(-1)^{2k} m^2((a)_{p,1})^2 k(n-1) \cdots (n-2k+1)}{k^{2k} ((b)_{q,1})^2 1! 1!} = \\ & = \frac{(-1)^{2k} m^2 k(n-1) \cdots (n-2k+1)}{k^{2k}} \left(\frac{(a)_{p,2}}{(b)_{q,2}} - \frac{((a)_{p,1})^2}{((b)_{q,1})^2} \right), \\ \alpha_{n-3k} &= \frac{(-1)^{3k} m^3 (a)_{p,3} 3k(n-1) \cdots (n-3k+1)}{k^{3k} (b)_{q,3} 3!} - \\ & - \frac{(-1)^{3k} m^3 k (a)_{p,1} (a)_{p,2} (n-1) \cdots (n-3k+1)}{k^{3k} (b)_{q,1} (b)_{q,2} 2!} - \\ & - \frac{(-1)^{3k} m^3 k (a)_{p,1} (n-1) \cdots (n-3k+1)}{k^{3k} (b)_{q,1} 1!} \left(\frac{(a)_{p,2}}{(b)_{q,2}} - \frac{((a)_{p,1})^2}{((b)_{q,1})^2} \right) = \\ & = \frac{(-1)^{3k} m^3 k(n-1) \cdots (n-3k+1)}{k^{3k}} \times \\ & \times \left(\frac{3(a)_{p,3}}{(b)_{q,3} 3!} - \frac{3(a)_{p,1} (a)_{p,2}}{2(b)_{q,1} (b)_{q,2}} + \frac{((a)_{p,1})^3}{((b)_{q,1})^3} \right), \end{aligned}$$

and so on. $\square$

In the case of the Gould-Hopper polynomials $k := m$, $m := (-1)^k h k^k$, and all of the $(a)_{p,i}$ and $(b)_{q,i}$ are absent. Therefore the powers of x less than $x - m$ are canceled and the recurrence equation is simplified.

Corollary 2.8. *Each Gould-Hopper polynomial $g_n^m(x, h)$ satisfies the following three-term recurrent equation*

$$g_n^m(x, h) - xg_{n-1}^m(x, h) - mh(n-1) \cdots (n-m+1)g_{n-m}^m(x, h) = 0.$$

From the latter recurrence together with (1.1), which, in turn, implies

$$D^k P_n(x) = n(n-1) \cdots (n-k+1)P_{n-k}(x),$$

it follows that each Gould-Hopper polynomial $g_n^m(x, h)$ satisfies the m -th order differential equation (see also [15])

$$ng_n^m(x, h) - x(g_n^m(x, h))' - hm(g_n^m(x, h))^{(m)} = 0.$$

Example 2.9. When $m = 2$ and $hm^m = -2$ that Gould-Hopper polynomials become the Hermite polynomials $H_n(x)$ with $mh = -1$, and the latter recurrence turns into well-known one for the Hermite polynomials ([16]):

$$H_n(x) - xH_{n-1}(x) + (n-1)H_{n-2}(x) = 0.$$

3. ANOTHER RECURRENT FORM OF THE CONNECTION PROBLEM SOLUTION FOR TWO DIFFERENT GENERALIZED HYPERGEOMETRIC APPELL POLYNOMIAL FAMILIES

Also, we admit that applying the same method of the equating of the powers of t we can specify the unknown coefficients α_k in formula (1.2) in another way.

Lemma 3.1. *Given two arbitrary power series $A(t)$ and $B(t)$ such that*

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \quad B(t) = \sum_{n=0}^{\infty} b_n t^n,$$

where $a_0 = b_0 = 1$, the coefficients of their ratio

$$\frac{A(t)}{B(t)} = \sum_{k=0}^{\infty} c_k t^k,$$

are expressed by the following recurrent formula

$$c_i = a_i - \sum_{j=0}^{i-1} b_{i-j} c_j, \quad \text{with } c_0 = 1.$$

Going back to the hypergeometric function we consider the transfer function $A_1(t)$ which is written in the form $A_1(t) = \sum_{n=0}^{\infty} a_n t^n$, of the generalized hypergeometric Appell polynomial $A_n^{(k)}(m, x)$. The values A_n of the corresponding generalized hypergeometric Appell polynomials $A_n^{(k)}(m, x)$ for

$x = 0$ and the coefficients a_n of its transfer function series $A_1(t)$ are related by the simple formula

$$A_n = a_n n!.$$

Applying Lemma 3.1, we obtain the recursive expressions for both the inverse problem and the explicit form of the generalized hypergeometric Appell polynomial $A_n^{(k)}(m, x)$ via the values of A_n .

Theorem 3.2. *The following identities hold*

$$\begin{aligned} \text{(i)} \quad A_n^{(k)}(m, x) &= \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{(n - ki)!} c_{ki} x^{n-ki}, \\ \text{(ii)} \quad x^n &= \sum_{i=0}^{\lfloor \frac{n}{k} \rfloor} \frac{n!}{(n - ki)!} c_{ki} A_{n-ki}^{(k)}(m, x), \end{aligned}$$

where $c_{ki} = - \sum_{j=0}^{i-1} \frac{A_{k(i-j)}}{(k(i-j))!} c_{kj}$, with $c_0 = 1$. In particular,

$$\begin{aligned} c_0 &= 1, & c_{3k} &= -a_{3k} + 2a_{2k}a_k - a_k^3, \\ c_k &= -a_k, & c_{4k} &= -a_{4k} + a_{2k}^2 + 2a_{3k}a_k - 3a_{2k}a_k^2 + a_k^4, \\ c_{2k} &= -a_{2k} + a_k^2, & \dots & \end{aligned}$$

For two different generalized hypergeometric functions $A(t)$ and $B(t)$ we obtain the following basic result.

Theorem 3.3. *Let $k_1, k_2 \in \mathbb{N}$ be such that $k_1 > k_2$ and $(k_1, k_2) = 1$. Then for two different generalized hypergeometric functions $A(t)$ and $B(t)$ such that*

$$\begin{aligned} A(t) &= {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| (-1)^{k_1} \frac{m_1 t^{k_1}}{k_1^{k_1}} \right] = \sum_{n=0}^{\infty} a_{k_1 n} t^{k_1 n}, \quad a_0 = 1, \\ B(t) &= {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_r \\ \beta_1, \dots, \beta_\rho \end{matrix} \middle| (-1)^{k_2} \frac{m_2 t^{k_2}}{k_2^{k_2}} \right] = \sum_{n=0}^{\infty} b_{k_2 n} t^{k_2 n}, \quad b_0 = 1, \end{aligned}$$

the coefficients of their ratio

$$\frac{A(t)}{B(t)} = \sum_{i=0}^{\infty} c_i t^i$$

are defined by the recurrence

$$c_0 = 1, \quad c_i = a_i - \sum_{j=0}^{i-1} b_{k_2 j} c_{i-k_2 j}.$$

Thus, applying this theorem for two different generalized hypergeometric Appell polynomial families, we obtain the following statement.

Corollary 3.4. *Given two different generalized hypergeometric Appell polynomial families $A_n^{(k_1)}(m_1, x)$ and $B_n^{(k_2)}(m_2, x)$ with the transfer functions $A(t)$ and $B(t)$, respectively, which is defined by the conditions of the Theorem 3.3, the solution of their connection problem has the form*

$$A_n^{(k_1)}(m_1, x) = \sum_{i=0}^n \frac{n!}{i!} c_{n-i} B_i^{(k_2)}(m_2, x), \quad (3.1)$$

where connective coefficients c_i are defined by Theorem 3.3, with $a_n = \frac{A_n}{n!}$, $b_n = \frac{B_n}{n!}$, and A_n, B_n being the values of the corresponding polynomials $A_n^{(k_1)}(m_1, x)$ and $B_n^{(k_2)}(m_2, x)$ at $x = 0$.

Example 3.5. In the case when $k_1 = 3$ and $k_2 = 2$ the first eight connection coefficients are given by the following formulas:

$$\begin{aligned} c_0 &= 1, & c_5 &= -b_2 a_3, \\ c_1 &= 0, & c_6 &= a_6 - b_6 + 2b_4 b_2 - b_2^3, \\ c_2 &= -b_2, & c_7 &= a_3(b_4 + b_2^2), \\ c_3 &= a_3, & c_8 &= -a_6 b_2 - b_8 + 2b_6 b_2 - 3b_4 b_2^2 + b_2^4 + b_4^2, \\ c_4 &= -b_4 + b_2^2, & \dots & \end{aligned}$$

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