Some applications of transversality for infinite dimensional manifolds

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Abstract. We present some transversality results for a category of Fréchet manifolds, the so-called MC^k -Fréchet manifolds. In this context, we apply the obtained transversality results to construct the degree of nonlinear Fredholm mappings by virtue of which we prove a rank theorem, an invariance of domain theorem and a version of Borsuk-Ulam type theorem.

Анотація. В роботі представлено деякі результати про трансверсальність для категорії MC^k -многовидів Фреше. Вони застосовуються до побудови степеня нелінійних відображень Фредгольма та доведення з його допомогою теорем про ранг, інваріантність області, а також версії теореми Улама-Борсука.

This paper is devoted to the development of transversality and its applications to degree theory of nonlinear Fredholm mappings for non-Banachable Fréchet manifolds. The elaboration is mostly, but not entirely, routine; we shall discuss the related issues.

In attempting to develop transversality and Fredholm theory to Fréchet manifolds we face the following drawbacks which are related to the lack of a suitable topology on a space of continuous linear mappings:

- (1) In general, the set of isomorphisms between Fréchet spaces is not open in the space of continuous linear mappings.
- (2) In general, the set of Fredholm operators between Fréchet spaces is not open in the space of continuous linear mappings.

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Also, a key point in the proof of an infinite dimensional version of Sard's theorem is that a Fredholm mapping φ near origin has a local representation of the form $\varphi(u, v) = (u, \eta(u, v))$ for some smooth mapping η ; indeed, this is a consequence of an inverse function theorem.

To obtain a version of Sard's theorem for Fréchet manifolds, based on the ideas of Müller [8], it was proposed by the author ([1]) to consider Fredholm operators which are Lipschitz on their domains. There is an appropriate metrizable topology on a space of Lipschitz linear mappings so that if we employ this space instead of a space of continuous linear mappings, the mentioned openness issues and the problem of stability of Fredholm mappings under small perturbation can be resolved. Furthermore, for mappings belong to a class of differentiability, bounded or MC^k -differentiability which is introduced in [8], a suitable version of an inverse function theorem is available, [8, Theorem 4.7] and [4, Proposition 7.1].

An example of Lipschitz-Fredholm mapping of class MC^k can be found in [6], where the Sard's theorem [1, Theorem 4.3] is applied to classify all the holomorphic functions locally definable; this gives the additional motivation to study further applications of Sard's theorem.

In this paper, we first improve the transversality theorem [2, Theorem 4.2] by considering all mappings of class MC^k , then use it to prove the parametric transversality theorem. Further, for Lipschitz-Fredholm mappings of class MC^k we apply the transversality theorem to construct the degree (due to Caccioppoli, Shvarts and Smale), which is defined as the group of non-oriented cobordism class of $\varphi^{-1}(q)$ for some regular value q .

Next, we prove a rank theorem for Lipschitz-Fredholm mappings of class MC^k , and use it to establish an invariance of domain theorem and a Fredholm alternative theorem. Also, using the parametric transversality theorem we obtain a Borsuk-Ulam type theorem.

1. BOUNDED FRÉCHET MANIFOLDS

In this section we will briefly recall the basics of MC^k -Fréchet manifolds for the convenience of readers, which also allows us to establish our notations for the rest of the paper. For more studies, we refer to [1, 2, 4].

Throughout the paper we assume that E, F are Fréchet spaces and we denote by $CL(E, F)$ the space of all continuous linear mappings from E to F topologized by the compact-open topology.

If T is a topological space by $U \subseteq T$ we mean U is open in T .

Let $\varphi : U \subseteq E \rightarrow F$ be a continuous map. If the directional (Gâteaux) derivatives

$$D\varphi(x)h = \lim_{t \rightarrow 0} \frac{\varphi(x + th) - \varphi(x)}{t}$$

exist for all $x \in U$ and all $h \in E$, and the induced map

$$D\varphi(x) : U \rightarrow CL(E, F)$$

is continuous for all $x \in U$, then we say that φ is a *Keller's differentiable map of class C_c^1* . The higher directional derivatives and C_c^k -mappings, $k \geq 2$, are defined in the obvious inductive fashion.

To define bounded or MC^k -differentiability, we endow a Fréchet space F with a translation invariant metric ϱ defining its topology, and then introduce the metric concepts which strongly depend on the choice of ϱ . We consider only metrics of the following form

$$\varrho(x, y) = \sup_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\|x - y\|_{F,n}}{1 + \|x - y\|_{F,n}},$$

where $\|\cdot\|_{F,n}$ is a collection of seminorms generating the topology of F .

Let σ be a metric that defines the topology of a Fréchet space E . Let $\mathbb{L}_{\sigma, \varrho}(E, F)$ be the set of all linear mappings $L : E \rightarrow F$ which are (globally) Lipschitz continuous as mappings between metric spaces E and F , that is

$$\mathcal{Lip}(L) := \sup_{x \in E \setminus \{0_F\}} \frac{\varrho(L(x), 0_F)}{\sigma(x, 0_F)} < \infty,$$

where $\mathcal{Lip}(L)$ is the (minimal) Lipschitz constant of L .

The translation invariant metric

$$\begin{aligned} d_{\sigma, \varrho} : \mathbb{L}_{\sigma, \varrho}(E, F) \times \mathbb{L}_{\sigma, \varrho}(E, F) &\longrightarrow [0, \infty), \\ (L, H) &\mapsto \mathcal{Lip}(L - H)_{\sigma, \varrho}, \end{aligned} \tag{1.1}$$

on $\mathbb{L}_{\sigma, \varrho}(E, F)$ turns it into an Abelian topological group. We always topologize the space $\mathbb{L}_{\sigma, \varrho}(E, F)$ by the metric (1.1).

Let $\varphi : U \subseteq E \rightarrow F$ be a continuous map. If φ is Keller's differentiable, $D\varphi(x) \in \mathbb{L}_{\sigma, \varrho}(E, F)$ for all $x \in U$ and the induced map

$$D\varphi(x) : U \rightarrow \mathbb{L}_{\sigma, \varrho}(E, F)$$

is continuous, then φ is called bounded differentiable or MC^1 and we write $\varphi^{(1)} = \varphi'$. We define for $k > 1$ mappings of class MC^k , recursively.

An MC^k -Fréchet manifold is a Hausdorff second countable topological space modeled on a Fréchet space with an atlas of coordinate charts such that the coordinate transition functions are all MC^k -mappings. We define MC^k -mappings between Fréchet manifolds as usual. Henceforth, we assume that M and N are connected MC^k -Fréchet manifolds modeled on Fréchet spaces (F, ϱ) and (E, σ) , respectively.

A mapping $\varphi \in \mathbb{L}_{\sigma, \varrho}(E, F)$ is called *Lipschitz-Fredholm operator* if its kernel has finite dimension and its image has finite co-dimension. The

index of φ is defined by

$$\text{Ind}\varphi = \dim(\ker \varphi) - \text{codim}(\text{Img } \varphi).$$

We denote by $\mathcal{LF}(E, F)$ the set of all Lipschitz-Fredholm operators, and by $\mathcal{LF}_l(E, F)$ the subset of $\mathcal{LF}(E, F)$ consisting of those operators of index l .

An MC^k -Lipschitz-Fredholm mapping $\varphi : M \rightarrow N$, $k \geq 1$, is a mapping such that for each $x \in M$ the derivative $D\varphi(x) : T_x M \rightarrow T_{f(x)} N$ is a Lipschitz-Fredholm operator. The index of φ , denoted by $\text{Ind}\varphi$, is defined to be the index of $D\varphi(x)$ for some x which does not depend on the choice of x , see [1, Definition 3.2].

Let $\varphi : M \rightarrow N$ be an MC^k -mapping, $k \geq 1$. We will denote by

$$T_x \varphi : T_x M \rightarrow T_{\varphi(x)} N$$

the tangent map of f at $x \in M$ from the tangent space $T_x M$ to the tangent space $T_{\varphi(x)} N$. Say that φ is an *immersion* (resp. *submersion*) whenever $T_x \varphi$ is injective (resp. surjective) and the range $\text{Img}(T_x \varphi)$ (resp. the kernel $\ker(T_x \varphi)$) splits in $T_{\varphi(x)} N$ (resp. $T_x M$) for any $x \in M$. An injective immersion $f : M \rightarrow N$ which gives an isomorphism onto a submanifold of N is called an *embedding*. A point $x \in M$ is called a *regular* point if $Df(x) : T_x M \rightarrow T_{f(x)} N$ is surjective. The corresponding value $f(x)$ is a regular value. Points and values other than regular are called critical points and values, respectively.

Let $\varphi : M \rightarrow N$ be an MC^k -mapping, $k \geq 1$. We say that φ is *transversal* to a submanifold $S \subseteq N$ and write $\varphi \pitchfork S$ if either $\varphi^{-1}(S) = \emptyset$, or if for each $x \in \varphi^{-1}(S)$

- (1) $(T_x \varphi)(T_x M) + T_{\varphi(x)} S = T_{\varphi(x)} N$, and
- (2) $(T_x \varphi)^{-1}(T_{\varphi(x)} S)$ splits in $T_x M$.

In terms of charts, $\varphi \pitchfork S$ whenever for each $x \in \varphi^{-1}(S)$ there exist charts $(\phi, \mathcal{U})$ around x and $(\psi, \mathcal{V})$ around $\varphi(x)$ such that

$$\psi : \mathcal{V} \rightarrow \mathcal{V}_1 \times \mathcal{V}_2$$

is an MC^k -isomorphism on a product, with

$$\psi(\varphi(x)) = (0_E, 0_E) \quad \varphi(S \cap \mathcal{V}) = \mathcal{V}_1 \times \{0_E\}.$$

Then the composite mapping

$$\mathcal{U} \xrightarrow{\varphi} \mathcal{V} \xrightarrow{\psi} \mathcal{V}_1 \times \mathcal{V}_2 \xrightarrow{\text{Pr}_{\mathcal{V}_2}} \mathcal{V}_2.$$

is an MC^k -submersion, where $\text{Pr}_{\mathcal{V}_2}$ is the projection onto $\mathcal{V}_2$.

2. TRANSVERSALITY THEOREMS

In this section we generalize [2, Theorem 4.2] and [2, Corollary 4.1] for not necessarily Lipschitz-Fredholm mappings and finite dimensional submanifolds. We will need the following version of the inverse function theorem for MC^k -mappings.

Theorem 2.1 ([8, Theorem 4.7]). *Let $\mathcal{U} \subseteq E$, $u_0 \in \mathcal{U}$ and $\varphi : \mathcal{U} \rightarrow E$ an MC^k -mapping, $k \geq 1$. If $\varphi'(u_0) \in \text{Aut}(E)$, then there exists $\mathcal{V} \subseteq \mathcal{U}$ of u_0 such that $\varphi(\mathcal{V})$ is open in E and $\varphi|_{\mathcal{V}} : \mathcal{V} \rightarrow \varphi(\mathcal{V})$ is an MC^k -diffeomorphism.*

Proposition 2.2. *Let $\varphi : M \rightarrow N$ be an MC^k -mapping, $S \subset N$ an MC^k -submanifold and $x \in \varphi^{-1}(S)$. Then $\varphi \pitchfork S$ if and only if there are charts $(\mathcal{U}, \phi)$ around x with $\phi(x) = 0_E$ and $(\mathcal{V}, \psi)$ around $y = \varphi(x)$ in S with $\psi(y) = 0_F$ such that the following hold:*

- (1) *There are subspaces $\mathbf{E}_1$ and $\mathbf{E}_2$ of E , and $\mathbf{F}_1$ and $\mathbf{F}_2$ of F such that $E = \mathbf{E}_1 \oplus \mathbf{E}_2$ and $F = \mathbf{F}_1 \oplus \mathbf{F}_2$. Moreover, $\psi(S \cap \mathcal{V}) = \mathbf{F}_1$ and*

$$\begin{aligned}\phi(\mathcal{U}) &= E_1 + E_2 \subseteq \mathbf{E}_1 \oplus \mathbf{E}_2 \\ \psi(\mathcal{V}) &= F_1 + F_2 \subseteq \mathbf{F}_1 \oplus \mathbf{F}_2,\end{aligned}$$

where $0_E \in E_i \subseteq \mathbf{E}_i$ and $0_F \in F_i \subseteq \mathbf{F}_i$, $i = 1, 2$.

- (2) *In the charts the local representative of φ has the form*

$$\varphi_{\phi\psi} = \bar{\varphi} + \hat{\varphi} \circ \text{Pr}_{E_2}, \quad (2.1)$$

where $\bar{\varphi} : E_1 + E_2 \rightarrow F_1$ is an MC^k -mapping, $\hat{\varphi}$ is an MC^k -isomorphism of $\mathbf{E}_2$ onto $\mathbf{F}_2$ and $\text{Pr}_{E_2} : E \rightarrow \mathbf{E}_2$ is the projection.

Proof. Sufficiency. Let $(\mathcal{U}, \phi)$ and $(\mathcal{V}, \psi)$ be charts that satisfy the assumptions we will prove $\varphi \pitchfork S$.

In the charts, by using the identifications $T_x M \simeq E$, $T_y \simeq F$, the tangent map $T_x \varphi : T_x M \rightarrow T_y N$ has the representation

$$T_x \varphi = \varphi'_{\phi\psi}(0_E) : E \rightarrow F.$$

Also, we have the identification $T_y S \simeq \mathbf{F}_1$.

Let $\text{Pr}_{F_2} : F \rightarrow \mathbf{F}_2$ be the projection onto $\mathbf{F}_2$. Since

$$\varphi'_{\phi\psi}(0_E) = (\bar{\varphi})' + \hat{\varphi} \circ \text{Pr}_{E_2}$$

and $(\bar{\varphi})'(0) : E \rightarrow \mathbf{F}_1$, it follows that for all $e \in E$ we have that $e = e_1 + e_2 \in \mathbf{E}_1 \oplus \mathbf{E}_2$

$$\varphi'_{\phi\psi}(0_E)e = (\bar{\varphi})'(0_E)e + \hat{\varphi} \circ \text{Pr}_{E_2}.$$

Thus, $\text{Pr}_{F_2} \circ \varphi'_{\phi\psi}(0_E)e = \text{Pr}_{F_2} \circ \hat{\varphi} \circ \text{Pr}_{F_2}(e)$ which means that

$$\text{Pr}_{F_2} \circ \varphi'_{\phi\psi}(0_E) = \text{Pr}_{F_2} \circ \hat{\varphi} \circ \text{Pr}_{F_2} \quad (2.2)$$

it is a surjective mapping of E onto $\mathbf{F}_2$.

Moreover, we have

$$\begin{aligned} \ker(\text{Pr}_{F_2} \circ \varphi'_{\phi\psi}(0_E)) &= \varphi'_{\phi\psi}(0_E)^{-1}(\text{Pr}_{F_2}(0_E)) = \varphi'_{\phi\psi}(0_E)^{-1}(F_1) \\ &= \{e = e_1 + e_2 \in \mathbf{E}_1 \oplus \mathbf{E}_2 \mid (\bar{\varphi})'(0_E)e + \hat{\varphi}(e_2) \in \mathbf{F}_1\} \\ &= \{e = e_1 + e_2 \in \mathbf{E}_1 \oplus \mathbf{E}_2 \mid \hat{\varphi}(e_2) = 0_F\} \\ &= \{e = e_1 + e_2 \in \mathbf{E}_1 \oplus \mathbf{E}_2 \mid e_2 = 0_E\} \\ &= \mathbf{E}_1. \end{aligned}$$

Which is an MC^k -splitting in E with a component E_2 . From (2.2) it follows that

$$\text{Pr}_{F_2} \circ \varphi'_{\phi\psi}(0_E) = \hat{\varphi} : \text{Pr}_{E_2} \rightarrow \text{Pr}_{F_2}$$

which is an MC^k -isomorphism.

Necessity. Suppose $\varphi \pitchfork S$. Since S is an MC^k -submanifold of N and $y = \varphi(x) \in S$, there is a chart $(\mathcal{W}, \mathfrak{w})$ around y having the submanifold property for S in N :

$$\begin{aligned} \mathfrak{w}(\mathcal{W}) &= W_1 + W_2 \subset \mathbf{F}_1 \oplus \mathbf{F}_2 = F, \\ \mathfrak{w}(S \cap \mathcal{W}) &= W_1 \subset F, \quad \mathfrak{w}(y) = 0_E. \end{aligned}$$

Also, there is a chart $(\mathcal{X}, \mathfrak{x})$ around x such that $\mathfrak{x}(x) = 0_F$, $\varphi(\mathcal{X}) \subset \mathcal{W}$ and

$$\varphi_{\mathfrak{x}\mathfrak{w}} : \mathfrak{x}(\mathcal{X}) \subseteq E \rightarrow \mathfrak{w}(\mathcal{W}) \subseteq F$$

is of class MC^k . It follows that

$$\varphi_{\mathfrak{x}\mathfrak{w}}(0_E) \circ \text{Pr}_{F_2} : E \rightarrow \mathbf{F}_2$$

is an MC^k -submersion as $\varphi \pitchfork S$. In other words, $\varphi_{\mathfrak{x}\mathfrak{w}} \circ \text{Pr}_{F_2}$ and $\mathbf{E}_1 := \varphi'_{\mathfrak{x}\mathfrak{w}}(0_E)^{-1}(\mathbf{F}_1)$ splits in E with the complement $\mathbf{E}_2$ such that

$$\eta := \text{Pr}_{F_2} \circ \varphi_{\mathfrak{x}\mathfrak{w}}(0_E) \mid_{\mathbf{E}_2} : \mathbf{E}_2 \rightarrow \mathbf{F}_2$$

is an MC^k -isomorphism. Consider the following map:

$$\tau := (\text{Pr}_{E_1} + \eta^{-1} \circ \text{Pr}_{F_2} \circ \varphi_{\mathfrak{x}\mathfrak{w}}) : \mathfrak{x}(\mathcal{X}) \rightarrow \mathfrak{w}(\mathcal{W}).$$

It follow that τ is an MC^k -mapping, $\tau(0_E) = 0_E$, and

$$\tau'(0_E) = (\text{Pr}_{E_1} + \eta^{-1} \circ \text{Pr}_{F_2} \circ \varphi_{\mathfrak{x}\mathfrak{w}})'(0_E) = \text{Pr}_{E_1} + \text{Pr}_{E_2} = \text{Id}_E,$$

since for all $e = e_1 + e_2 \in \mathbf{E}_1 \oplus \mathbf{E}_2$ we have that

$$(\varphi_{\mathfrak{x}\mathfrak{w}})'(0_E)e = (\varphi_{\mathfrak{x}\mathfrak{w}})'(0_E)e_1 + (\varphi_{\mathfrak{x}\mathfrak{w}})'(0_E)e_2.$$

Whence,

$$\text{Pr}_{F_2} \circ (\varphi_{\mathbf{xw}})'(0_E)e = \text{Pr}_{F_2} \circ (\varphi_{\mathbf{xw}})'(0_E)e_2 = \tau(e_2),$$

and therefore

$$\tau^{-1} \circ \text{Pr}_{F_2} \circ (\varphi_{\mathbf{xw}})'(0_E)e = \tau^{-1}(\tau(e_2)) = \text{Pr}_{E_2}.$$

By the inverse mapping theorem 2.1, τ is a local MC^k -diffeomorphism. Assume $0_E \in \mathcal{X}_1 \subseteq \mathbf{x}(\mathcal{X})$ is small enough. Let also

$$\mathbf{x}_1 : \mathcal{X}_1 \rightarrow 0_E \in \mathcal{X}_2 \subseteq E$$

be an MC^k -diffeomorphism such that

$$\tau \circ \mathbf{x}_1^{-1} = \text{Id}_F. \quad (2.3)$$

Thus,

$$\text{Pr}_{F_2} \circ \tau \circ \varphi_{\mathbf{xw}} = \eta \circ \text{Pr}_{E_2}. \quad (2.4)$$

If, $e = e_1 + e_2 \in \mathbf{x}_1(\mathcal{X}_1)$ and $\mathbf{x}_1^{-1}(e) = \bar{e}_1 + \bar{e}_2$, then by (2.3) and (2.4) we obtain

$$\begin{aligned} \tau \circ \mathbf{x}_1^{-1}(e) &= \tau(\bar{e}_1 + \bar{e}_2) \\ &= \bar{e}_1 + \eta^{-1} \circ \text{Pr}_{F_2} \circ \varphi_{\mathbf{xw}}(\bar{e}_1 + \bar{e}_2) \\ &= e_1 + e_2. \end{aligned}$$

Therefore, $\bar{e}_1 = e_1$ and $\tau^{-1} \circ \text{Pr}_{F_2} \circ \varphi_{\mathbf{xw}} = e_2$ and so

$$\text{Pr}_{F_2} \circ \varphi_{\mathbf{xw}}(\bar{e}_1 + \bar{e}_2) = \tau(e_2) = \tau \circ \text{Pr}_{E_2}(e_1 + e_2).$$

This means that

$$\text{Pr}_{E_2} \circ \varphi_{\mathbf{xw}} \circ \mathbf{x}_1^{-1}(e) = \eta \circ \text{Pr}_{E_2}(e)$$

for all $e \in \mathbf{x}_1(\mathcal{X}_1)$.

Now, define

$$\begin{aligned} \phi &:= \mathbf{x}_1 \circ \mathbf{x}, & \mathcal{U} &:= \mathbf{x}^{-1}(\mathcal{X}), \\ \psi &:= \mathbf{w}, & \mathcal{V} &:= \text{small enough neighborhood of } y \text{ in } \mathcal{W}. \end{aligned}$$

Then, $(\phi, \mathcal{U})$ and $(\psi, \mathcal{V})$ are the desired charts. Indeed,

$$\begin{aligned} \varphi_{\phi\psi} &= \mathbf{w} \circ \varphi \circ (\mathbf{x}_1 \circ \mathbf{x}) \\ &= \mathbf{w} \circ \varphi \circ \mathbf{x}^{-1} \circ \mathbf{x}_1^{-1} = \varphi_{\mathbf{xw}} \circ \mathbf{x}_1^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \varphi_{\phi\psi} &= \text{Pr}_{F_1} \circ \varphi_{\mathbf{xw}} \circ \mathbf{x}_1^{-1} + \text{Pr}_{F_2} \circ \varphi_{\mathbf{xw}} \circ \mathbf{x}_1^{-1} \\ &= \bar{\varphi} + \hat{\varphi} \circ \text{Pr}_{E_2}, \end{aligned}$$

whenever we set $\hat{\varphi} := \eta$ and $\bar{\varphi} := \text{Pr}_{E_1} \circ \varphi_{\mathbf{xw}} \circ \mathbf{x}_1^{-1}$. □

Theorem 2.3 (Transversality Theorem). *Let $\varphi : M \rightarrow N$ be an MC^k -mapping, $k \geq 1$, $S \subset N$ an MC^k -submanifold, and $\varphi \pitchfork S$. Then $\varphi^{-1}(S)$ is either empty of MC^k -submanifold of M with*

$$(T_x\varphi)^{-1}(T_yS) = T_x(\varphi^{-1}(S)), \quad x \in \varphi^{-1}(S), \quad y = \varphi(x).$$

If S has finite co-dimension in N , then $\text{codim}(\varphi^{-1}(S)) = \text{codim}S$. Moreover, if $\dim S = m < \infty$ and φ is an MC^k -Lipschitz-Fredholm mapping of index l , then $\dim \varphi^{-1}(S) = l + m$.

Proof. Let $x \in \varphi^{-1}(S)$. Then by Proposition 2.2 there are charts $(\phi, \mathcal{U})$ around x and $(\psi, \mathcal{V})$ around $y = \varphi(x)$ such that

$$\begin{aligned} \phi(\mathcal{U}) &= E_1 + E_2 \subseteq \mathbf{E}_1 \oplus \mathbf{E}_2, \\ \psi(\mathcal{V}) &= F_1 + F_2 \subseteq \mathbf{F}_1 \oplus \mathbf{F}_2, \\ \varphi_{\phi\psi} &= \overline{\varphi} + \hat{\varphi} \circ \text{Pr}_{E_2}, \end{aligned} \tag{2.5}$$

where $\overline{\varphi} : E_1 + E_2 \rightarrow F_1$ is an MC^k -mapping, $\hat{\varphi}$ is an MC^k -isomorphism of $\mathbf{E}_2$ onto $\mathbf{F}_2$, and $\text{Pr}_{E_2} : E \rightarrow \mathbf{E}_2$ is the projection.

Let $\hat{e} \in \varphi^{-1}(S) \cap \mathcal{U}$. Then $\hat{f} = \varphi(\hat{e}) \in S \cap \mathcal{V}$ and $\psi(\varphi(\hat{e})) \in F_1 \subset \mathbf{F}_1$. Due to (2.5), if $\phi(\hat{e}) = e_1 + e_2 \in E_1 + E_2$, then

$$\begin{aligned} \varphi_{\phi\psi}(\phi(\hat{e})) &= \varphi_{\phi\psi}(e_1 + e_2) \\ &= \overline{\varphi}(e_1 + e_2) + \hat{\varphi}(e_2) \in F_1 \subset \mathbf{F}_1. \end{aligned}$$

Therefore $\hat{\varphi}(e_2) = 0_E$ and $e_2 = 0_E$, since $\hat{\varphi}(e_2) \in \mathbf{F}_2$ and $\mathbf{F}_1 \cap \mathbf{F}_2 = \{0_F\}$.

Thus, $\phi(\hat{e}) \in F_1$ for all $\hat{e} \in \varphi^{-1}(S) \cap \mathcal{U}$. Hence,

$$E_1 \subset \phi(\varphi^{-1}(S) \cap \mathcal{U}),$$

since for each $e_1 \in E_1$ we have that

$$\varphi_{\phi\psi}(e_1) = \overline{\varphi}(e_1) + \hat{\varphi} \circ \text{Pr}_{E_2}(e_1) = \overline{\varphi}(e_1) \in F_1.$$

Hence, $\psi \circ \varphi \circ \phi^{-1}(e_1) \in F_1$ implies that

$$\varphi \circ \phi^{-1}(e_1) \in \psi^{-1}(F_1) = S \cap \mathcal{V}$$

and so $\varphi \circ \phi^{-1}(e_1) \in S$ which means $\phi^{-1}(e_1) \in \varphi^{-1}(S) \cap \mathcal{V}$ and that yields $e_1 \in \phi(\varphi^{-1}(S) \cap \mathcal{V})$. Therefore, for each $x \in \varphi^{-1}(S)$ there is a chart $(\phi, \mathcal{U})$ with

$$\phi(\mathcal{U}) = E_1 + E_2 \subset \mathbf{E}_1 \oplus \mathbf{E}_2,$$

$\phi(x) = 0_E$, and $\phi(\varphi^{-1}(S) \cap \mathcal{V}) = E_1$, which means that $\varphi^{-1}(S)$ is an MC^k -submanifold in M .

In the charts, we have that

$$T_x \simeq E, \quad T_y N \simeq F, \quad T_x(\varphi^{-1}(S)) \simeq \mathbf{E}_1, \quad T_y S \simeq \mathbf{F}_1.$$

Then from the proof of Proposition 2.2 we also get that

$$\varphi'_{\phi\psi}(0_E)^{-1}(\mathbf{F}_1) = \mathbf{E}_1,$$

which yields $(T_x\varphi)^{-1}(T_yS) = T_x(\varphi^{-1}(S))$.

If S has finite co-dimension then $\mathbf{F}_2$ has finite dimension and thus by Proposition 2.2,

$$\text{codim}(\varphi^{-1}(S)) = \text{codim}(\varphi^{-1}(S \cap \mathcal{V})) = \dim(\mathbf{F}_2) = \text{codim}(S).$$

The proof of the last statement is standard. $\square$

As an immediate consequence we obtain:

Corollary 2.4. *Let $\varphi : M \rightarrow N$ be an MC^k -mapping, $k \geq 1$. If q is a regular value of φ , then the level set $\varphi^{-1}(q)$ is a submanifold of M and its tangent space at $p = \varphi(q)$ is $\ker T_p\varphi$. Moreover, if q is a regular value of φ and φ is an MC^k -Lipschitz-Fredholm mapping of index l , then $\dim(\varphi^{-1}(S)) = l$.*

To prove the parametric transversality theorem we apply the following Sard's theorem.

Theorem 2.5 ([2, Theorem 3.2]). *If $\varphi : M \rightarrow N$ is an MC^k -Lipschitz-Fredholm map with $k > \max\{\text{Ind}\varphi, 0\}$. Then, the set of regular values of φ is residual in N .*

Theorem 2.6 (The Parametric Transversality Theorem). *Let A be a manifold of dimension n , $S \subset N$ a submanifold of finite co-dimension m . Let also $\varphi : M \times A \rightarrow N$ be an MC^k -mapping, $k \geq \{1, n - m\}$. If $\varphi \pitchfork S$, then the set of all points $x \in M$ such that the mappings*

$$\varphi_x : A \rightarrow N, (\varphi_x(\cdot) := \varphi(x, \cdot))$$

are transversal to S , is residual in M .

Proof. Let $\mathbf{S} = \varphi^{-1}(S)$, $\text{Pr}_M : M \times A \rightarrow M$ the projection onto M and $\text{Pr}_\mathbf{S}$ be its restriction to $\mathbf{S}$. First we prove that $\text{Pr}_\mathbf{S}$ is an MC^k -Fredholm-Lipschitz mapping of index $n - m$, i.e.,

$$T_{(m,a)}\text{Pr}_\mathbf{S} : T_{(m,a)}\mathbf{S} \rightarrow T_mM$$

is a Lipschitz-Fredholm operator of index $n - m$.

By Theorem 2.3 the inverse image $\mathbf{S}$ is an MC^k -submanifold of $M \times A$, with model space $\mathbb{S}$, so that $\text{Pr}_\mathbf{S}$ is an MC^k -mapping.

Let π_M and $\pi_\mathbf{S}$ be the local representatives of Pr_M and $\text{Pr}_\mathbf{S}$, respectively. We show that π_M and consequently $\pi_\mathbf{S}$ are Lipschitz-Fredholm operators of

index $n - m$. Finite dimensionality of $\mathbb{R}^n$ and closedness of $\mathbb{S}$ implies that

$$K := \mathbb{S} + (\{0\} \times \mathbb{R}^n)$$

is closed in $E \times \mathbb{R}^n$. Also, $\text{codim} K$ is finite because it contains the finite co-dimensional subspace $\mathbb{S}$. Therefore K has a finite-dimensional complement $K_1 \subset E \times \{0\}$, that is $E \times \mathbb{R}^n = K \oplus K_1$. Let $K_2 := \mathbb{S} \cap \{0\} \times \mathbb{R}^n$. Since $K_2 \subset \mathbb{R}^n$ we can choose closed subspaces $\mathbb{S}_1 \subset \mathbb{R}^n$ and $\mathbb{R}_0 \subset \{0\} \times \mathbb{R}^n$ such that $\mathbb{S} = \mathbb{S}_1 \oplus K_1$ and $\{0\} \times \mathbb{R}^n = K_1 \oplus \mathbb{R}_0$. Whence,

$$K = \mathbb{S}_1 \oplus K_1 \oplus \mathbb{R}_0, \quad E \times \mathbb{R}^n = \mathbb{S}_1 \oplus K_1 \oplus \mathbb{R}_0 \oplus K_2.$$

The mapping $\pi_{\mathbb{S}}|_{\mathbb{S}_1 \oplus K_2}: \mathbb{S}_1 \oplus K_2 \rightarrow E$ is an isomorphism, $K_1 = \ker \pi_{\mathbb{S}}$, and $\pi_M(K_2)$ is a finite dimensional complement to $\pi_M(\mathbb{S})$ in $\mathbb{R}^n$. Thus, π_M is a Lipschitz-Fredholm operator and we have that

$$\begin{aligned} \text{Ind} \pi_M &= \dim K_1 - \dim K_2 \\ &= \dim(K_1 \oplus \mathbb{R}_0) - \dim(\mathbb{R}_0 \oplus K_2). \end{aligned}$$

Since, $K_1 \oplus \mathbb{R}_0 = \{0\} \times \mathbb{R}^n$ and $\mathbb{R}_0 \oplus K_2$ is a complement to $\mathbb{S}$ in $E \times \mathbb{R}^n$ and therefore its dimension is n , so the index of π_M is $n - m$.

Now, we prove that if x is a regular value of $\text{Pr}_{\mathbb{S}}$ if and only if $\varphi_x \pitchfork S$. From the definition of $\varphi \pitchfork$ we have $\forall (x, a) \in \mathbf{S}$

$$(T_{(x,a)}\varphi)(T_x M \times T_a A) + T_{\varphi(x,a)}S = T_{\varphi(x,a)}N, \quad (2.6)$$

and

$$(T_{(x,a)}\varphi)^{-1}(T_{\varphi(x,a)}S) \text{ splits in } T_x M \times T_a A. \quad (2.7)$$

Since A has finite dimension, it follows that the mapping $a \in A \mapsto \varphi(x, a)$ for a fixed $x \in M$ is transversal to S if and only if

$$\forall (x, a) \in \mathbf{S}, T_a \varphi_x(T_a A) + T_{\varphi(x,a)}S = T_{\varphi(x,a)}S. \quad (2.8)$$

Since $\text{Pr}_{\mathbb{S}}$ is a Lipschitz-Fredholm mapping, $\ker T\text{Pr}_{\mathbb{S}}$ splits at any point as its dimension is finite. Hence x is a regular value of $\text{Pr}_{\mathbb{S}}$ if and only if

$$\forall (x, a) \in \mathbf{S}, \forall v \in T_x M, \exists u \in T_a A: T_{(v,u)}\varphi(v, u) \in T_{(x,a)}S. \quad (2.9)$$

Pick $x \in M$ and $a \in A$ such that $(x, a) \in \mathbf{S}$ and let $w \in T_{(x,a)}S$. By (2.6) and (2.7) we obtain that there exist $v \in T_x M$, $x_1 \in T_x M$, $y_1 \in T_{(x,a)}S$ such that

$$T_{(x,a)}\varphi(v, x_1) + y_1 = w. \quad (2.10)$$

Then, there exists $x_2 \in T_x M$ such that $T_{(x,a)}\varphi(v, x_2) \in T_{\varphi(x,a)}S$. Hence,

$$\begin{aligned} w &= T_{(x,a)}\varphi(v, x_1) - T_{(x,a)}\varphi(v, x_2) + T_{(x,a)}\varphi(v, x_2) + y_1 \\ &= T_{(x,a)}\varphi(0, x_1 - x_2) + T_{(x,a)}\varphi(v, x_2) + y_1 \\ &= T_{(x,a)}\varphi(0, u) + T_{\varphi(x,a)}S + y_2 \in T_a \varphi_x(T_a A), \end{aligned}$$

where $u = x_1 - x_2$ and $y_2 = T_{(x,a)}\varphi(v, x_2) + y_1 \in T_{\varphi(x,a)}S$. Thus, (2.8) holds.

Now we show that (2.8) implies (2.9). Pick $a \in A$, $x \in M$ such that $(x, a) \in \mathbf{S}$. Let $v \in T_x M$, $a_1 \in T_a A$, $y_1 \in T_{\varphi(x,a)} S$ and set

$$w := T_{(x,a)}\varphi(v, x_1) + y_1.$$

By (2.8) there exist $a_2 \in T_a A$ and $y_2 \in T_{\varphi(x,a)} S$ such that

$$w = T_a \varphi_x(a_2) + y_2.$$

Then,

$$\begin{aligned} 0_E &= T_{(x,a)}\varphi(v, a_1) - T_a \varphi_x(a_2) + y_1 - y_2 \\ &= T_{(x,a)}\varphi(v, a_1 - a_2) + y_1 - y_2, \end{aligned}$$

so

$$T_{(x,a)}\varphi(v, a_1 - a_2) = y_2 - y_1 \in T_{\varphi(x,a)} S,$$

and thus (2.9) holds. Thus, we showed that if x is a regular value of Pr_S if and only if $\varphi_x \pitchfork S$. Since $\text{Pr}_S : \mathbf{S} \rightarrow M$ is a Lipschitz-Fredholm of class MC^k with the index $n - m$ and

$$\text{codim } \mathbf{S} = \text{codim } S = m$$

and $k > \{0, n - m\}$, the Sard's theorem 2.5 concludes the theorem. $\square$

3. THE DEGREE OF LIPSCHITZ-FREDHOLM MAPPINGS

In this section we construct the degree of MC^k -Lipschitz-Fredholm mappings and apply it to prove an invariance of domain theorem, a rank theorem and a Borsuk-Ulam type theorem. The construction of the degree relies on the following transversality result.

Theorem 3.1 ([2, Theorem 3.3]). *Let $\varphi : M \rightarrow N$ be an MC^k -Lipschitz-Fredholm mapping, $k \geq 1$. Let also $\iota : \mathcal{A} \rightarrow N$ be an MC^1 -embedding of a finite dimension manifold $\mathcal{A}$ with $k > \max\{\text{Ind}\varphi + \dim \mathcal{A}, 0\}$. Then there exists an MC^1 fine approximation $\mathbf{g}$ of ι such that $\mathbf{g}$ is embedding and $\varphi \pitchfork \mathbf{g}$. Moreover, suppose S is a closed subset of $\mathcal{A}$ and $\varphi \pitchfork \iota(S)$, then $\mathbf{g}$ can be chosen so that $\iota = \mathbf{g}$ on S .*

We shall need the following theorem that gives the connection between proper and closed mappings.

Theorem 3.2 ([9, Theorem 1.1]). *Let A, B be Hausdorff manifolds, where A is a connected infinite dimensional Fréchet manifold, and B satisfies the first countability axiom, and let $\varphi : A \rightarrow B$ be a continuous closed non-constant map. Then φ is proper.*

Let $\varphi : M \rightarrow N$ be a non-constant closed Lipschitz-Fredholm mapping with index $l \geq 0$ of class MC^k such that $k > l + 1$. If q is a regular value of φ , then by Theorem 3.2 and Corollary 2.4 the preimage $\varphi^{-1}(q)$ is a compact submanifold of dimension l .

Let $\iota : [0, 1] \hookrightarrow N$ be an MC^1 -embedding that connects two distinct regular values q_1 and q_2 . By Theorem 3.1 we may suppose ι is transversal to φ . Thus, by Theorem 2.3 the preimage $\mathbf{M} := \varphi^{-1}(\iota([0, 1]))$ is a compact $(l + 1)$ -dimensional submanifold of M such that its boundary, $\partial\mathbf{M}$, is the disjoint union of $\varphi^{-1}(q_1)$ and $\varphi^{-1}(q_2)$, $\partial\mathbf{M} = \varphi^{-1}(q_1) \amalg \varphi^{-1}(q_2)$. Therefore, $\varphi^{-1}(q_1)$ and $\varphi^{-1}(q_2)$ are non-oriented cobordant which gives the invariance of the mapping. Following Smale [10] we associate to φ a degree, denoted by $\deg \varphi$, defined as the non-oriented cobordism class of $\varphi^{-1}(q)$ for some regular value q . If $l = 0$, then $\deg \varphi \in \mathbb{Z}_2$ is the number modulo 2 of preimage of a regular value.

Let $\mathcal{O} \subseteq M$. Suppose $\varphi : \overline{\mathcal{O}} \rightarrow N$ is a non-constant closed continuous mapping such that its restriction to $\mathcal{O}$ is an MC^{k+1} -Lipschitz-Fredholm mapping of index k , $k \geq 0$. Let $p \in N \setminus \varphi(\partial\overline{\mathcal{O}})$ and let $\mathbf{p}$ a regular value of φ in the connected component of $N \setminus \varphi(\partial\overline{\mathcal{O}})$ containing p , the existence of such regular value follows from Sard's theorem 2.5. Again, we associate to φ a degree, $\deg(\varphi, p)$, defined as non-oriented class of k -dimensional compact manifold $\varphi^{-1}(\mathbf{p})$. This degree does not depend on the choice of $\mathbf{p}$.

The following theorem which presents the local representation of MC^k -mappings is crucial for the rest of the paper.

Theorem 3.3 ([1, Theorem 4.2]). *Let $\varphi : \mathcal{U} \subseteq E \rightarrow F$ be an MC^k -mapping, $k \geq 1$, $u_0 \in \mathcal{U}$. Suppose that $D\varphi(u_0)$ has closed split image $\mathbf{F}_1$ with closed topological complement $\mathbf{F}_2$ and split kernel $\mathbf{E}_2$ with closed topological complement $\mathbf{E}_1$. Then, there are two open sets $\mathcal{U}_1 \subseteq \mathcal{U}$ and $\mathcal{V} \subseteq \mathbf{F}_1 \oplus \mathbf{E}_2$ and an MC^k -diffeomorphism $\Psi : \mathcal{V} \rightarrow \mathcal{U}_1$, such that*

$$(\varphi \circ \Psi)(f, e) = (f, \eta(f, e))$$

for all $(f, e) \in \mathcal{V}$, where $\eta : \mathcal{V} \rightarrow \mathbf{E}_2$ is an MC^k -mapping.

Theorem 3.4 (Rank theorem for MC^k -mappings). *Let $\varphi : \mathcal{U} \subseteq E \rightarrow F$ be an MC^k -mapping, $k \geq 1$. Suppose $u_0 \in \mathcal{U}$ and $D\varphi(u_0)$ has closed split image $\mathbf{F}_1$ with closed complement $\mathbf{F}_2$ and split kernel $\mathbf{E}_2$ with closed complement $\mathbf{E}_1$. Also, assume*

$$D\varphi(u)|_{\mathbf{E}_1} : \mathbf{E}_1 \rightarrow D\varphi(u)(E)$$

is an MC^k -isomorphism for each $u \in \mathcal{U}$ and $D\varphi(\mathcal{U})(E)$ is closed in F . Then, there exist open sets

$$\mathcal{U}_1 \subseteq \mathbf{F}_1 \oplus \mathbf{E}_2, \quad \mathcal{U}_2 \subseteq E, \quad \mathcal{V}_1 \subseteq F, \quad \mathcal{V}_2 \subseteq F,$$

and MC^k -diffeomorphisms $\phi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ and $\psi : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ such that

$$(\phi \circ \varphi \circ \psi)(f, e) = (f, 0), \quad \forall (f, e) \in \mathcal{U}_1.$$

Proof. By Theorem 3.3 there exists an MC^k -diffeomorphism

$$\psi : \mathcal{U}_1 \subseteq \mathbf{F}_1 \oplus \mathbf{E}_2 \rightarrow \mathcal{U}_2 \subseteq E$$

such that

$$\varphi(f, e) = (\varphi \circ \psi)(f, e) = (f, \eta(f, e)),$$

where $\eta : \mathcal{V} \rightarrow \mathbf{E}_2$ is an MC^k -mapping. Let $\text{Pr}_1 : F \rightarrow \mathbf{F}_1$ be the projection. Then $\text{Pr}_1 \circ D\varphi(f, e)(w, v) = (w, 0)$, for $w \in \mathbf{F}_1$ and $v \in \mathbf{E}_2$ since

$$D\varphi(f, e)(w, v) = (w, D\eta(f, e)(w, v)).$$

Hence, $\text{Pr}_1 \circ D\varphi(f, e)|_{\mathbf{F}_1 \times \{0\}}$ is the identity mapping, $\text{Id}_{\mathbf{F}_1}$, on $\mathbf{F}_1$. Thereby,

$$D\varphi(f, e)|_{\mathbf{F}_1 \times \{0\}} : \mathbf{F}_1 \times \{0\} \rightarrow D\varphi(f, e)(\mathbf{F}_1 \oplus \mathbf{E}_2)$$

is one-to-one and therefore by our assumption

$$D\varphi(f, e) \circ \text{Pr}_1|_{D\varphi(f, e)(\mathbf{F}_1 \oplus \mathbf{E}_2)}$$

is the identity mapping. Suppose

$$(w, D\eta(f, e)(w, v)) \in D\varphi(f, e)(\mathbf{F}_1 \oplus \mathbf{E}_2).$$

Then we obtain $D\eta(f, e)v = 0$ for all $v \in \mathbf{E}_2$, which means that $D_2\eta(f, e) = 0$, since

$$\begin{aligned} (D\varphi(f, e) \circ \text{Pr}_1)(w, D\eta(f, e)(w, v)) &= D\varphi(f, e)(w, 0) \\ &= (w, D\eta(f, e)(w, 0)) \\ &= (w, D_1(f, e)w). \end{aligned}$$

Therefore

$$D^2\varphi(f, e)v = (0, D_2\eta(f, e)v),$$

i.e., $D^2\varphi(f, e) = 0$ which means that φ does not depend on the variable $y \in \mathbf{E}_2$. Let $\text{Pr}_2 : \mathbf{F}_1 \oplus \mathbf{E}_2 \rightarrow \mathbf{F}_1$ be the projection and

$$\varphi_f := \varphi(f, e) = (\varphi \circ \text{Pr}_2)(f, e),$$

so that

$$\varphi_f : \text{Pr}_2(\mathcal{U}_1) \subset \mathbf{F}_1 \rightarrow F.$$

Let also $\mathbf{v}_0 := (\text{Pr}_2 \circ \psi^{-1})(u_0)$ and $\mathcal{V} \subseteq \mathcal{U}$ be an open neighborhood of $\mathbf{v}_0$. Define the following mapping

$$\Phi : \mathcal{V} \times \mathbf{F}_2 \rightarrow \mathbf{F}_1 \oplus \mathbf{F}_2, \quad \Phi(f, e) = \varphi(f) + (0, e).$$

By the open mapping theorem

$$D\Phi(\mathbf{v}_0, 0) = (D\varphi(\mathbf{v}_0), \text{Id}_{\mathbf{F}_2}) : E \oplus \mathbf{F}_2 \rightarrow F$$

is a linear MC^k -isomorphism, where $\text{Id}_{\mathbf{F}_2}$ is the identity mapping of $\mathbf{F}_2$. Now Φ satisfies the inverse function theorem 2.1 at $\mathbf{v}_0$, whence there exist $\mathcal{V}_1, \mathcal{V}_2 \subseteq F$ such that $(\mathbf{v}_0, 0) \in \mathcal{V}_2$ and $\Phi(\mathbf{v}_0, 0) = \varphi_{\mathbf{v}_0}(\mathbf{v}_0) \in \mathcal{V}_1$ and an MC^k -diffeomorphism $\phi : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ such that $\phi^{-1} = \Phi|_{\mathcal{V}_1}$. Thus, for $(f, 0) \in \mathcal{V}_2$ we have

$$(\phi \circ \varphi)(f) = (\varphi \circ \Phi)(f, 0) = (f, 0),$$

and therefore,

$$(\phi \circ \varphi \circ \psi)(f, e) = (f, 0), \quad \forall (f, e) \in \mathcal{U}_1. \quad \square$$

As an immediate consequence we get the following:

Corollary 3.5 (Rank theorem for Lipschitz-Fredholm mappings). *Let*

$$\varphi : M \rightarrow N$$

be an MC^∞ -Lipschitz-Fredholm mapping of index k and

$$\dim \ker D\varphi(x) = m, \quad \forall x \in M.$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the topological complements of $\mathbb{R}^m$ in E and $\mathbb{R}^{m-k}$ in F , respectively. Then, there exist charts

$$\phi : \mathcal{U} \subseteq M \rightarrow E = \mathbb{R}^m \oplus \mathcal{C}_1 \quad \text{with} \quad \phi(x) = 0_E,$$

$$\psi : \mathcal{V} \subseteq N \rightarrow F = \mathbb{R}^{m-k} \oplus \mathcal{C}_2 \quad \text{with} \quad \psi(x) = 0_F,$$

such that

$$\psi \circ \varphi \circ \phi^{-1}(f, 0) = (f, 0).$$

A version of a constant rank theorem for mappings from manifolds modeled on locally convex spaces to finite dimensional manifolds is obtained in [5]. Also, in the case of Banach manifolds a constant rank theorem can be found in [7] and [11].

The following theorem gives the openness property of the set of Lipschitz-Fredholm mappings.

Theorem 3.6 ([1, Theorem 3.2]). *The set $\mathcal{LF}(E, F)$ is open in $\mathcal{L}_{d,g}(E, F)$ with respect to the topology defined by the metric (1.1). Furthermore, the function $T \rightarrow \text{Ind}T$ is continuous on $\mathcal{LF}(E, F)$, and hence it is constant on connected components of $\mathcal{LF}(E, F)$.*

The proof of the following theorem is a modification of [11, Theorem 2].

Theorem 3.7. *Let $\varphi : M \rightarrow N$ be a Lipschitz-Fredholm mapping of class MC^k , $k \geq 1$. Then the set*

$$\text{Sing}(\varphi) := \{m \mid D\varphi(m) \text{ is not injective}\}$$

is nowhere dense in M .

Proof. This is a local problem so we assume M is an open set in E and N is an open set in F . Let $s \in \text{Sing}(\varphi)$ be arbitrary and $\mathcal{U}$ an open neighborhood of s in $\text{Sing}(\varphi)$. For each $n \in \mathbb{N} \cup \{0\}$ define

$$S_n := \{m \in M \mid \dim D\varphi(m) \geq n\}.$$

Then $M = M_0 \supset M_1 \supset \cdots$, whence there exists a unique n_0 such that

$$M = M_{n_0} \neq M_{n_0+1}.$$

Let $m_0 \in M_{n_0} \setminus M_{n_0+1}$ be such that $\dim \ker D\varphi(m_0) = n_0$. By Theorem 3.6, there exists an open neighborhood $\mathcal{V}$ of m_0 in $\mathcal{U}$ such that for all $v \in \mathcal{V}$ we have that $\dim \ker D\varphi(v) \leq n_0$ and hence $\dim D\varphi(v) = n_0 \geq 1$. By Corollary 3.5, there is a local representative φ around zero such that

$$\psi \circ \varphi \circ \phi^{-1}(f, e) = (f, 0)$$

for $(f, e) \in \mathbb{C}_1 \oplus \mathbb{R}^{n_0}$, which contradicts the injectivity of φ . Therefore, $\text{Sing}(\varphi)$ contains a nonempty open set. The closedness of $\text{Sing}(\varphi)$ is obvious in virtue of Theorem 3.6. $\square$

Theorem 3.8 (Invariance of domain for Lipschitz-Fredholm mappings). *Let $\varphi : M \rightarrow N$ be an MC^k -Lipschitz-Fredholm mapping of index zero, $k > 1$. If φ is locally injective, then φ is open.*

Proof. Let $p \in U \subseteq M$ and $q = \varphi(p)$. The point p has a connected open neighborhood $\mathcal{U} \subseteq M$ such that $\varphi|_{\overline{\mathcal{U}}} : \overline{\mathcal{U}} \rightarrow N$ is proper and injective. Whence $q \notin \varphi(\partial\mathcal{U})$ and $\varphi(\partial\mathcal{U})$ is closed in N . Let $\mathcal{V}$ be a connected component of $N \setminus \varphi(\partial\mathcal{U})$ containing q which is its open neighborhood. Since $\mathcal{U}$ is connected, it follows that $\varphi(\mathcal{U}) \subset \mathcal{V}$. Then we get from $\varphi(\partial\mathcal{U}) \cap \mathcal{V} = \emptyset$ that $\overline{\mathcal{U}} \cap \varphi^{-1}(\mathcal{V}) = \mathcal{U}$ and so $\varphi|_{\mathcal{U}} : \mathcal{U} \rightarrow N$ is proper and injective. By Theorem 3.7 there is a point $x \in M$ such that the tangent map $T_x\varphi$ is injective and since $\text{Ind}\varphi = 0$ it is surjective too. Therefore, $y = \varphi(x)$ is a regular value with $\varphi^{-1}(y) = \{x\}$ and so $\deg \varphi = 1$. It follows that φ is surjective, because otherwise any point from $N \setminus \varphi(M)$ is regular and thus $\deg \varphi = 0$ which is contradiction. Hence $\mathcal{V} = \varphi(\mathcal{U})$ is an open neighborhood of q . $\square$

An analogue of this theorem for Banach manifolds can be found in [11].

Corollary 3.9 (Nonlinear Fredholm alternative). *Let $\varphi : M \rightarrow N$ be an MC^k -Lipschitz-Fredholm mapping of index zero, $k > 1$. If N is connected and φ is locally injective, then φ is surjective and finite covering mapping. If M is connected and N is simply connected, then φ is a homeomorphism.*

Proof. The proof is standard. $\square$

The following theorem is a generalization of the Borsuk-Ulam theorem. Its proof is a slight modification of the Banach case, cf. [3].

Theorem 3.10. *Let $\varphi : \overline{\mathcal{U}} \rightarrow F$ be a non-constant closed Lipschitz-Fredholm mapping of class MC^2 with index zero, where $U \subseteq F$ is symmetric. If φ is odd and for $u_0 \in \overline{U}$ we have $u_0 \notin \varphi(\partial\overline{\mathcal{U}})$. Then $\deg(\varphi, u_0) \equiv 1 \pmod{2}$.*

Proof. Since $D\varphi(u_0)$ is a Lipschitz-Fredholm mapping with index zero

$$F = \mathbf{F}_1 \oplus \ker \varphi = \mathbf{F}_2 \oplus \text{Im} \varphi$$

and $\dim \mathbf{F}_2 = \dim \ker \varphi$. The image $\varphi(\partial\mathcal{U})$ is closed as φ is closed, hence

$$\mathbf{a} = \varrho(\varphi(\mathcal{U}), u_0) > 0$$

because $u_0 \notin \varphi(\partial\mathcal{U})$.

Let $\phi : F \rightarrow F$ be a global Lipschitz-compact linear operator with $\text{Lip}(\phi) < \mathbf{b}$ for some $\mathbf{b} > 0$. Define the mapping $\Phi_\phi : \overline{U} \rightarrow F$ by $\Phi_\phi(u) = \varphi(u) + \phi(u)$. Then Φ_ϕ is a Lipschitz-Fredholm mapping of index zero. Suppose $\mathbf{b} < \mathbf{a}/\mathbf{k}$ for some $\mathbf{k} > 1$, then

$$\varrho(\Phi_\phi(u), u_0) \geq \varrho(\varphi(u), u_0) - \text{Lip}(\phi)\varrho(U, u_0) > \mathbf{a} - \mathbf{b}\mathbf{k} > 0, \quad \forall u \in \partial\mathcal{U}.$$

Therefore, $u_0 \notin \Phi_\phi(\partial\mathcal{U})$. Hence $\deg(\varphi, u_0) = \deg(\Phi_\phi, u_0)$ as the mapping

$$\psi : [0, 1] \times \overline{U} \rightarrow F, \quad \psi(t, u) = \varphi(u) + t\phi(u),$$

is proper and $u_0 \notin \psi(\partial U)$ for all t . Considering the fact that

$$\psi(-u) = -\psi(u),$$

we may use the perturbation by compact operators to find the degree of φ . Let $\mathbf{C}$ be the set of global Lipschitz-compact linear operators $\phi : F \rightarrow F$ with $\text{Lip}(\phi) < \mathbf{b} < \mathbf{a}/\mathbf{k}$. Let also $\hat{\phi} \in \mathbf{C}$ be such that its restriction to $\mathbf{F}_1$ equals u_0 and

$$\hat{\phi}|_{\ker D\varphi(u_0)} : \ker D\varphi(u_0) \rightarrow \mathbf{F}_2$$

is an MC^1 -isomorphism. Then $D\varphi(u_0) + \hat{\phi}$ and consequently $D\varphi(u_0)$ is an MC^1 -isomorphism. Now define the mapping

$$\Psi : \mathcal{U} \times \mathbf{C} \rightarrow F, \quad \Psi(u, \phi) = \Phi_\phi(u).$$

Then for a sufficiently small $\mathbf{b}$ the differential

$$D\Psi(u, \phi)(v, \psi) = (D\varphi(u) + \phi)v + \psi(u)$$

is surjective at u_0 as $D\varphi(u_0)$ is an MC^1 -isomorphism. It is also clear that it is surjective at the other points. Then, the mapping Ψ satisfies the assumption of Theorem 2.6. Therefore, $\Psi^{-1}(u_0)$ is a submanifold and the mapping $\Pi : \Psi^{-1}(u_0) \rightarrow \mathbf{C}$ induced by the projection onto the second order is Lipschitz-Fredholm of index zero. By employing the local version of Sard's theorem we may find a regular point $\bar{\phi}$ of Π , and from the proof of the Theorem 2.6 it follows that u_0 is a regular value of $\Phi_{\bar{\phi}}$. Hence u_0 is a

regular value of φ . Thus, the properness and the identity $\varphi(-u) = -\varphi(u)$ imply that

$$\varphi^{-1}(u_0) = \{u_0, f_1, -f_1, \dots, f_m, -f_m\}$$

and therefore $\deg(\varphi, u_0) \equiv 1 \pmod{2}$. $\square$

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