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# Special semi-reducible pseudo-Riemannian spaces

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**Abstract.** The paper contains necessary conditions allowing to reduce matrix tensors of pseudo-Riemannian spaces to special forms called semi-reducible, under assumption that the tensor defining tensor characteristic of semireducibility spaces, is idempotent. The tensor characteristic is reduced to the spaces of constant curvature, Ricci-symmetric spaces and conformally flat pseudo-Riemannian spaces.

The obtained results can be applied for construction of examples of spaces belonging to special types of pseudo-Riemannian spaces.

The research is carried out locally in tensor shape, without limitations imposed on a sign of a metric.

**Анотація.** В роботі досліджується можливість зведення метричного тензора псевдоріманових просторів до спеціального виду, який називають напівзвідністю. Тензорна ознака спрощується для просторів сталої кривини, Річчі-симетричних просторів та конформно-плоских псевдоріманових просторів. Отримані умови носять необхідний характер і базуються на ідемпотентності тензора, що задає тензорну ознаку напівзвідності простору. В роботі також вивчаються деякі властивості напівзвідних псевдоріманових просторів.

Для напівзведення метрики псевдоріманового простору необхідно і достатньо виконання умов алгебраїчного та диференціального характеру. Ці умови називаються тензорною ознакою напівзвідності. Досліджуються умови диференціювання та їх продовження для тензорної ознаки. Зокрема, доведено, що вектор із тензорної ознаки напівзвідності лежить в ядрі тензора з цієї ж ознаки.

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*Ключові слова*: псевдо-ріманові простори, напівзвідні простори, спеціальні псевдо-ріманові простори.

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Для спеціальних просторів, просторів сталої кривини, Річчі-симетричних просторів та конформно-плоских просторів знайдені умови, яким задовольняють ці вектор та тензор за необхідністю. При отриманні результатів використовувалась властивість ідемпотентності тензора з ознаки напівзвідності. Умова ідемпотентності важлива для напівзведення просторів без вимоги знаковизначення метричного тензора.

Тому отримані результати цікаві для вивчення просторів з умовою ідемпотентності. Як відомо, умова ідемпотентності може бути послаблена і замінена вимогою, щоб матриця тензора $b_{ij}$ мала прості елементарні дільники та дійсні корені. В такому вигляді умова наводиться в книзі Л. Ейзенхарта «Ріманова геометрія», але без вимоги існування дійсних коренів.

Актуальним залишається питання вивчення достатніх умов, а також умов глобального чи топологічного характеру. Це дозволить ефективно досліджувати геометричні властивості, як узагальнених просторів, так і конкретних просторів загальної теорії відносності.

Результати можуть бути застосовані при побудові прикладів просторів, що належать до спеціальних типів псевдоріманових просторів.

Дослідження ведуться локально, в тензорній формі, без обмежень на знак метрики.

## 1. Introduction

Semi-reducible decomposition of the metric of a pseudo-Riemannian space $V_n$, $(n > 2)$, with a metric tensor $g_{ij}$ is a representation of the following form

$$ds^2 = ds_1^2(x^1, x^2, \dots, x^r) + \sigma(x^1, x^2, \dots, x^r)\, ds_2^2(x^{r+1}, x^{r+2}, \dots, x^n).$$

Here $ds_1^2$ and $ds_2^2$ are independent metrics defined in different coordinates and the function $\sigma$ depends on the coordinates $ds_1^2$ only.

A space $V_n$ that permits at least one semi-reducible decomposition is called a *semi-reducible space.*

The following extreme cases of decomposition are paid a special attention

$$ds^2 = \left(dx^1\right)^2 + \rho(x_1) ds_1^2(x^2, x^3, \dots, x^n) \tag{1.1}$$

and

$$ds^2 = ds_1^2(x^1, x^2, \dots, x^{n-1}) + \rho(x^1, x^2, \dots, x^{n-1}) \left(dx^n\right)^2. \tag{1.2}$$

Pseudo-Riemannian spaces, where the equation (1.1) holds, are called *equidistant.* Equation (1.1) is a necessary and sufficient condition for existence of concircular vector field in $V_n$ $[1, 10]$.

Condition (1.2) can be treated as a generalization of the static metric in general theory of relativity.

## 2. Tensor characteristic of semi-reducible spaces

A pseudo-Riemannian space $V_n$ can be represented as a semi-reducible space if and only if there exists a symmetrical idempotent tensor which is not proportional to metric in this space.

The latter tensor should correspond to the following limitation

$$b_{\alpha i} b^{\alpha}_{j} = b_{ij}, \tag{2.1}$$

$$b_{ij,k} = u_i b_{jk} + u_j b_{ik}. \tag{2.2}$$

Here $u_i = u_{,i} = \partial_i u$ are some gradient vector, comma is a sign of a covariant derivative, and $b^i_j = b_{\alpha j} g^{\alpha i}$, $g^{ij}$ are elements of the inverse matrix to $g_{ij}$.

Equations (2.1) and (2.2) are called *tensor characteristic* of semi-reducibility of pseudo-Riemannian spaces. This paper treats semi-reducible spaces, which differ from reducible spaces, namely by the fact that $u_i \neq 0$.

Tensor characteristic of semi-reducible spaces was found by G. I. Kruchkovich [10, 11].

Diferentiating (2.1) and taking into account (2.2), we can get

$$u_\alpha b^{\alpha}_{j} b_{ik} + u_\alpha b^{\alpha}_{i} b_{jk} = 0. \tag{2.3}$$

Alternating by indices $j, k$ also gives

$$u_\alpha b^{\alpha}_{j} b_{ik} - u_\alpha b^{\alpha}_{k} b_{ij} = 0.$$

Let us re-assign indices $i$ and $k$

$$u_\alpha b^{\alpha}_{j} b_{ki} - u_\alpha b^{\alpha}_{i} b_{kj} = 0. \tag{2.4}$$

Then adding (2.4) and (2.3) up, we can clearly see, that

$$u_\alpha b^{\alpha}_{j} = 0, \tag{2.5}$$

which implies

$$u_{\alpha,i} b^{\alpha}_{j} = -u_\alpha u^\alpha b_{ij}, \tag{2.6}$$

where $u^i = u_\alpha g^{\alpha i}$.

Taking into account the Ricci identity, integrability conditions for (2.2) can be written down as follows:

$$\begin{aligned} b_{\alpha i} R^{\alpha}_{jkl} + b_{\alpha j} R^{\alpha}_{ikl} = (u_{l,i} - u_l u_i) b_{jk} + \\ + (u_{l,j} - u_l u_j) b_{ik} - (u_{k,i} - u_k u_i) b_{jl} - (u_{k,j} - u_k u_j) b_{il}. \end{aligned} \tag{2.7}$$

Here $R^h_{ijk}$ is a Riemann tensor for $V_n$.

Wrapping (2.7) and substituting (2.5) and (2.6), we move to the expression

$$(u_{l,i} - u_l u_i) b = (u^{\alpha}_{\alpha,} - u_\alpha u^\alpha) b_{il} + b_{\alpha i} R^{\alpha}_{l} - b_{\alpha\beta} R^{\alpha\beta}_{.il.}, \tag{2.8}$$

where $b = b_{\alpha\beta} g^{\alpha\beta}$, $u^{\alpha}_{\alpha,} = u_{\alpha,\beta} g^{\alpha\beta}$, $R^{hj}_{il} = R^h_{il\beta} g^{\alpha j}$ [1, 8].

Thus, we have proved the following statement

**Theorem 2.1.** *Vector $u_i$ in the tensor characteristic of semi-reducible pseudo-Riemannian spaces conforms to the conditions:* (2.5), (2.6), (2.8).

Furthermore, wrapping (2.2) by indices $i, j$ and taking into account (2.5), we get

$$b_{,i} = 0.$$

Multiplying (2.8) by $b^i_j$ and wrapping by $i$, taking into account (2.1), we obtain

$$u^{\alpha}_{\alpha,}b_{ij} + b_{\alpha i}R^{\alpha}_{j} - b_{\alpha\beta}b^{\gamma}_{i}R^{\alpha\beta}_{\gamma j} = 0. \tag{2.9}$$

Applying the obtained results we can formulate some expressions, which will be useful in the further discussion.

Let us wrap (2.8) with $u_i$ and $u_l$. Then we obtain respectively

$$u^{\beta}b_{\alpha j}R^{\alpha}_{\beta kl} = u^{\alpha}(u_{l,\alpha} - u_{\alpha}u_l)b_{jk} - u^{\alpha}(u_{k,\alpha} - u_{\alpha}u_k)b_{jl};$$

$$b_{\alpha i}R^{\alpha}_{jk\beta}u^{\beta} + b_{\alpha j}R^{\alpha}_{ik\beta}u^{\beta} = \\ = u^{\alpha}(u_{\alpha,i} - u_{\alpha}u_i)b_{jk} + u^{\alpha}(u_{\alpha,j} - u_{\alpha}u_j)b_{ik}.$$

Let us turn our attention to the direct research on semi-reducibility of special spaces.

## 3. Semi-reducible spaces of constant curvature

Pseudo-Riemannian spaces $V_n$ are called *spaces of constant curvature*, when the Riemann tensor $V_n$ complies to the condition

$$R^h_{ijk} = \frac{R}{n(n-1)}\left(\delta^h_k g_{ij} - \delta^h_j g_{ik}\right), \tag{3.1}$$

here $\delta^h_k$ are Kronecker symbols, and $R$ is a scalar curvature of

$$R = R_{\alpha\beta}g^{\alpha\beta},$$

see [14, 15].

Substitute (3.1) in the integrability conditions of the tensor characteristic of semi-reducible spaces. Then, the following expression arises:

$$\tau_{li}b_{jk} + \tau_{lj}b_{ik} - \tau_{ki}b_{jl} - \tau_{kj}b_{il} = 0. \tag{3.2}$$

Here

$$\tau_{ij} = u_{i,j} - u_i u_j + \frac{R}{n(n-1)}g_{ij}. \tag{3.3}$$

Let us alternate (3.2) by indices $i, k$:

$$\tau_{li}b_{jk} - \tau_{lk}b_{ji} - \tau_{kj}b_{il} + \tau_{ij}b_{kl} = 0,$$

then re-assign indices $k$ and $j$:

$$\tau_{li}b_{jk} - \tau_{lj}b_{ki} - \tau_{kj}b_{il} + \tau_{ik}b_{jl} = 0, \tag{3.4}$$

and add (3.4) and (3.2):

$$\tau_{li}b_{jk} - \tau_{jk}b_{il} = 0.$$

Let us also wrap the latter with $g^{jk}$:

$$b\tau_{li} = \tau b_{li},$$

where $\tau = \tau_{\alpha\beta}g^{\alpha\beta}$.

Taking into account (3.3) we get

$$b\left(u_{i,j} - u_i u_j + \frac{R}{n(n-1)}g_{ij}\right) = \tau b_{ij}. \tag{3.5}$$

Let us calculate the covariant derivative of the expression (3.5):

$$b\left(u_{i,jk} - u_{i,k}u_j - u_i u_{j,k}\right) = \tau_{,k}b_{ij} + \tau u_i b_{jk} + \tau u_j b_{ik}. \tag{3.6}$$

Taking into account (3.5) in the equation (3.6):

$$bu_{i,jk} - \tau b_{ik}u_j - \tau b_{jk}u_i - 2bu_i u_k u_j + \\ + \frac{Rb}{n(n-1)}u_j g_{ij} + \frac{Rb}{n(n-1)}u_i g_{ij} = \\ = \tau_{,k}b_{ij} + \tau u_i b_{jk} + \tau u_j b_{ik}.$$

Let us alternate by indices $j, k$

$$bu_\alpha R^\alpha_{ijk} = \frac{Rb}{n(n-1)}\left(u_k g_{ij} - u_j g_{ik}\right) + b_{ij}(\tau_{,k} - 2\tau u_k) - b_{ik}(\tau_{,j} - 2\tau u_j),$$

and substitute (3.1) into the latter formula:

$$b_{ij}(\tau_{,k} - 2\tau u_k) - b_{ik}(\tau_{,j} - 2\tau u_j) = 0. \tag{3.7}$$

Multiplying (3.5) by $b^i_k$ and wrapping by index $i$, we get

$$b\left(\frac{R}{n(n-1)} - u_\alpha u^\alpha\right) = \tau,$$

which implies

$$\tau_{,i} = -2bu^\alpha u_{\alpha,i}.$$

Hence (3.7) reduces to the following form:

$$\rho_k b_{ij} - \rho_j b_{ik} = 0, \tag{3.8}$$

where

$$\rho_k = bu^\alpha u_{\alpha,k} + \tau u_k.$$

Let us wrap (3.8) with $g^{ij}$:

$$b\rho_k = \rho_\alpha b^\alpha_k.$$

Now using the latter identity, let us also wrap (3.8) with $b^k_m$ and re-assign indices $k$ and $m$:

$$b\rho_k b_{ij} - \rho_j b_{ik} = 0. \tag{3.9}$$

By subtracting (3.8) and(3.9), we get $b = 1$ or $\rho_k = 0$.

Suppose that $\rho_k \neq 0$. Then we can choose a vector $\xi^k$ in such a way that: $\rho_\alpha \xi^\alpha = 1$ and taking into account (3.8) we get

$$b_{ij} = \gamma \rho_i \rho_j, \tag{3.10}$$

where $\gamma$ is some invariant, $\gamma \rho_\alpha \rho^\alpha = 1$ [4,5].

**Theorem 3.1.** *Conditions* (3.10) *or* $bu^\alpha u_{\alpha,k} + \tau u_k = 0$ *hold in semi-reducible spaces of constant curvature.*

## 4. Semi-reducible Ricci-symmetric spaces

Pseudo-Riemannian spaces $V_n$ are called *Ricci-symmetric*, whenever the the Ricci tensor of $V_n$ satisfies the following identity [6,9]:

$$R_{ij,k} = 0. \tag{4.1}$$

Symmetrizing the equation (2.8) we get:

$$b_{\alpha i} R^\alpha_j - b_{\alpha j} R^\alpha_i = 0. \tag{4.2}$$

The latter, together with (2.2), implies

$$u_\alpha R^\alpha_j b_{ik} + u_i b_{\alpha k} R^\alpha_j + b_{\alpha i} R^\alpha_{j,k} - u_\alpha R^\alpha_i b_{jk} - u_j b_{\alpha k} R^\alpha_i - b_{\alpha j} R^\alpha_{i,k} = 0. \tag{4.3}$$

Multiplying (4.2) by $u^j$ and wrapping by index $j$, we obtian

$$b_{\alpha i} R^\alpha_\beta u^\beta = 0.$$

Equaton (4.3), when (4.1) and (4.2) are taking into consideration, can be re-written as follows:

$$u_\alpha R^\alpha_j b_{ik} + u_i b_{\alpha k} R^\alpha_j - u_\alpha R^\alpha_i b_{jk} - u_j b_{\alpha i} R^\alpha_k = 0. \tag{4.4}$$

Multiplying (4.4) by $b^i_m$, and wrapping it by index $i$ and re-assigning $i$ and $m$ we obtain:

$$u_\alpha R^\alpha_j b_{ik} - u_j b_{\alpha i} R^\alpha_k = 0. \tag{4.5}$$

As far as $b_{ij} \neq 0$ and $u_i \neq 0$, (4.5) implies

$$u_\alpha R^\alpha_i = \overset{1}{\tau} u_i, \tag{4.6}$$

and

$$b_{\alpha i} R^\alpha_j = \overset{2}{\tau} b_{ij}. \tag{4.7}$$

Here $\overset{1}{\tau}, \overset{2}{\tau}$ are some invariants.

Differentiating (4.7), we get

$$\overset{1}{\tau} u_j b_{ik} = \overset{2}{\tau}_k b_{ij} + \overset{2}{\tau} u_j b_{ik}. \tag{4.8}$$

Multiplying (4.8) by $b^j_m$, wrapping by $j$ we can see that $\overset{2}{\tau}_k = 0$. Then equation (4.8) transforms into the following identity:

$$\left(\overset{1}{\tau} - \overset{2}{\tau}\right) u_j b_{ik} = 0,$$

whence $\overset{1}{\tau} = \overset{2}{\tau} = const$. Thus, the following statement is true

**Theorem 4.1.** *Conditions* (4.6), (4.7) *hold in semi-reducible Ricci-symmetric spaces, while* $\overset{1}{\tau} = \overset{2}{\tau}$ *are constant.*

**Corollary 4.2.** *Vector* $u_i$ *complies to the following conditions in the semi-reducible Ricci-symmetric spaces*

$$b\,(u_{i,j} - u_i u_j) = \left(u^{\alpha}{}_{,\alpha} - u_\alpha u^\alpha + \overset{1}{\tau}\right) b_{ij} - b_{\alpha\beta} R^{\alpha}{}_{ij}{}^{\beta}. \tag{4.9}$$

It is easy to see that (4.9) is true, if we substitute (4.7) in (2.8).

The Ricci identity for the vector $u_i$ implies:

$$u_{i,jk} - u_{i,kj} = u_\alpha R^\alpha_{ijk}.$$

We can prove the following corollary.

**Corollary 4.3.** *Vector* $u_i$ *complies to the following conditions in semi-reducible Ricci-symmetric spaces*

$$u^{\alpha}{}_{,\alpha k} - u^{\alpha}{}_{,k\alpha} = \overset{1}{\tau} u_i.$$

Spaces whose Ricci tensor satisfies the following conditions

$$R_{ij} = \frac{R}{n} g_{ij},$$

are called *Einstein spaces* [7, 9].

For Einstein spaces the scalar curvature $R$ is always constant. Einstein spaces are particular cases of Ricci-symmetric spaces. Therfefore the following statement hold:

**Corollary 4.4.** *In semi-reducible Einstein spaces* $\overset{1}{\tau} = \dfrac{R}{n}$.

## 5. Semi-reducible conformal flat spaces

A necessary and sufficient condition for a pseudo-Riemannian space to be defined as a conformal flat space is his compliance to the following conditions [3, 13]

$$R_{hijk} = P_{hk} g_{ij} - P_{hj} g_{ik} + P_{ij} g_{hk} - P_{ik} g_{hj}, \tag{5.1}$$

$$P_{ij,k} - P_{ik,j} = 0, \tag{5.2}$$

where

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right). \tag{5.3}$$

Taking into account (5.3) we can write (4.2) in the following form

$$b_{\alpha i}P^{\alpha}_{j} - b_{\alpha j}P^{\alpha}_{i} = 0,$$

where $P^h_i = P_{\alpha i}g^{\alpha h}$.

Substituting (5.1) in (2.7), we get

$$\begin{aligned} b_{\alpha i}P^{\alpha}_{l}g_{jk} - b_{\alpha i}P^{\alpha}_{k}g_{jl} + b_{\alpha j}P^{\alpha}_{l}g_{ik} - b_{\alpha j}P^{\alpha}_{k}g_{il} = \\ = b_{jk}\lambda_{li} + b_{ik}\lambda_{lj} - b_{li}\lambda_{jk} - b_{lj}\lambda_{ik}, \end{aligned} \tag{5.4}$$

where

$$\lambda_{ij} = u_{i,j} - u_i u_j + P_{ij}. \tag{5.5}$$

Alternating (5.4) by indices $j$ and $l$

$$\begin{aligned} b_{\alpha i}P^{\alpha}_{l}g_{jk} - b_{\alpha i}P^{\alpha}_{j}g_{lk} - b_{\alpha j}P^{\alpha}_{k}g_{il} + b_{\alpha l}P^{\alpha}_{k}g_{ij} = \\ = b_{jk}\lambda_{li} - b_{lk}\lambda_{ji} - b_{li}\lambda_{jk} + b_{ji}\lambda_{lk}, \end{aligned}$$

and re-assigning indices $i$ and $l$ we obtain:

$$\begin{aligned} b_{\alpha l}P^{\alpha}_{i}g_{jk} - b_{\alpha l}P^{\alpha}_{j}g_{ik} - b_{\alpha j}P^{\alpha}_{k}g_{li} + b_{\alpha i}P^{\alpha}_{k}g_{li} = \\ = b_{jk}\lambda_{il} - b_{ik}\lambda_{jl} - b_{il}\lambda_{jk} + b_{jl}\lambda_{ik}. \end{aligned} \tag{5.6}$$

Summing up (5.6) and (5.4), we pass to

$$b_{\alpha i}P^{\alpha}_{l}g_{jk} - b_{\alpha j}P^{\alpha}_{k}g_{il} = b_{jk}\lambda_{li} - b_{il}\lambda_{jk}. \tag{5.7}$$

Further, wrap the latter equation by $j, k$:

$$b_{\alpha i}P^{\alpha}_{l} = \frac{b}{n}\lambda_{li} - \frac{\lambda}{n}b_{il} + \frac{\check{P}}{n}g_{il}, \tag{5.8}$$

where $\check{P} = b^{\alpha}_{\beta}P^{\beta}_{\alpha}$.

Multiplying (5.8) by vector $u^i$ and wrapping it by index $i$ we get:

$$bu^{\alpha}\lambda_{\alpha l} + \check{P}u_l = 0. \tag{5.9}$$

Multiplying (5.7) by $bu^j$ and wrapping it by index $j$, taking into account (5.9), we obtaiin

$$bb_{\alpha i}P^{\alpha}_{l}u_k = \check{P}u_k b_{il},$$

which can also be written as follows:

$$bb_{\alpha i}P^{\alpha}_{l} = \check{P}b_{il}.$$

Hence one can write (5.8) in the following form:

$$b^2\lambda_{ji} - \left(b\lambda + n\check{P}\right)b_{ij} + b\check{P}g_{ij} = 0.$$

Substituting (5.5) into the latter identity we get

$$b^2(u_{i,j} - u_i u_j) + b^2 P_{ij} + b\check{P} g_{ij} - \left(b\lambda + n\check{P}\right) b_{ij} = 0. \tag{5.10}$$

Now multiply (5.10) by $b^i_k$ and wrap by index $i$

$$\left(-b^2 u^\alpha u_\alpha + 2P\check{P} - b\lambda - n\check{P}\right) b_{jk} = 0.$$

Hence

$$\check{P} = \frac{b\left((b-1)u^\alpha u_\alpha + u^\alpha{}_{,\alpha} + P\right)}{2b - n}.$$

Therefore (5.10) can be written as follows

$$u_{i,j} - u_i u_j + P_{ij} + \mu g_{ij} - \overset{3}{\tau} b_{ij} = 0, \tag{5.11}$$

where $\mu = \dfrac{\check{P}}{b}$ and $\overset{3}{\tau} = -\left(b\lambda + n\check{P}\right)\dfrac{1}{b^2}$.

Differentiating (5.11), we get

$$u_{i,jk} - u_{i,k}u_j - u_i u_{j,k} + P_{ij,k} + \mu_k g_{ij} + + \overset{3}{\tau}_k b_{ij} + \overset{3}{\tau} b_{ik} u_j + \overset{3}{\tau} b_{jk} u_i = 0, \tag{5.12}$$

where $\mu_k = \mu_{,k} = \partial_k \mu$; $\overset{3}{\tau}_k = \overset{3}{\tau}_{,k} = \partial_k \overset{3}{\tau}$.

Alternating (5.12) and taking into account the Ricci identity, we obtain

$$u_\alpha R^\alpha_{ijk} - u_{i,k}u_j + u_{i,j}u_k + \mu_k g_{ij} - \mu_j g_{ik} + \\ + \left(\overset{3}{\tau}_k - \overset{3}{\tau} u_k\right) b_{ij} - \left(\overset{3}{\tau}_j - \overset{3}{\tau} u_j\right) b_{ik} = 0.$$

Taking into account(5.11) and (5.1)

$$\left(u_\alpha P^\alpha_k - \mu u_k + \mu_k\right) g_{ij} - \left(u_\alpha P^\alpha_j - \mu u_j + \mu_j\right) g_{ik} + \\ + \left(\overset{3}{\tau}_k - 2\overset{3}{\tau} u_k\right) b_{ij} - \left(\overset{3}{\tau}_j - 2\overset{3}{\tau} u_j\right) b_{ik} = 0. \tag{5.13}$$

Multiply (5.13) by vector $u^i$, and wrap by index $i$:

$$\left(u_\alpha P^\alpha_k - \mu u_k + \mu_k\right) u_j - \left(u_\alpha P^\alpha_j - \mu u_j + \mu_j\right) u_k = 0.$$

Asssuming that $u_i \not\equiv 0$, choose a vector $\xi^i$ so that $\xi^\alpha u_\alpha = 1$. Then

$$u_\alpha P^\alpha_k - \mu u_k + \mu_k = \overset{4}{\tau} u_k, \tag{5.14}$$

where $\overset{4}{\tau} = \left(u_\alpha P^\alpha_\beta - \mu u_\beta + \mu_\beta\right)\xi^\beta$.

Substituting (5.14) into (5.13), we get

$$\overset{4}{\tau} u_k g_{ij} - \overset{4}{\tau} u_j g_{ik} + \left(\overset{3}{\tau}_k - 2\overset{3}{\tau} u_k\right) b_{ij} - \left(\overset{3}{\tau}_j - 2\overset{3}{\tau} u_j\right) b_{ik} = 0. \tag{5.15}$$

Multiplying by $b^k_l$, wrapping by index $k$ and re-assigning $l$ by $k$:

$$b^\alpha_k \overset{3}{\tau}_\alpha b_{ij} - \left(\overset{3}{\tau}_j - 2\overset{3}{\tau} u_j + \overset{4}{\tau} u_j\right) b_{ik} = 0. \tag{5.16}$$

Wrapping (5.16) by indices $i$ and $k$, we obtain

$$b_j^\alpha \overset{3}{\tau}_\alpha - b\left(\overset{3}{\tau}_j - 2\overset{3}{\tau}u_j + \overset{4}{\tau}u_j\right) = 0. \tag{5.17}$$

Wrapping further (5.16) by indices $i$ and $j$, we also obtain

$$bb_k^\alpha \overset{3}{\tau}_\alpha - \overset{3}{\tau}_\alpha b_k^\alpha = 0. \tag{5.18}$$

Then (5.18) and (5.17) imply that $b = 1$ and equations (5.17) can be written as follows:

$$b_j^\alpha \overset{3}{\tau}_\alpha = \overset{3}{\tau}_j - 2\overset{3}{\tau}u_j + \overset{4}{\tau}u_j. \tag{5.19}$$

Hence (5.16) takes the following form:

$$b_k^\alpha \overset{3}{\tau}_\alpha b_{ij} = b_j^\alpha \overset{3}{\tau}_\alpha b_{ik}.$$

Wrapping (5.15) by indices $i$ and $j$, taking into account (5.19), we see that

$$\overset{4}{\tau} = 0.$$

Equations (5.14) can be transformed into the following form:

$$u_\alpha P_k^\alpha = \mu u_k - \mu_k. \tag{5.20}$$

Furthermore, equation (5.15) means that

$$\overset{3}{\tau}_i = 2u_i \overset{3}{\tau}.$$

**Theorem 5.1.** *If a conformal flat space permits a semi-reducible decomposition, then vector $u_i$ complies to equations* (5.11)*, invariant $\mu$* (5.20)*, and $\overset{3}{\tau}$ is defined by formula $\overset{3}{\tau} = e^{2u}$.*

Thus, we have found the conditions of differential nature, which constraint the vector $u_i$ and parameters of obtained differential equation [12, 16, 17].

## 6. Conclusion

The paper treats some characteristics of semi-reducible pseudo-Riemannian spaces. Certain algebraic and differential conditions are necessary and sufficient in order to define the metric of pseudo-Riemannian spaces as semi-reducible. These conditions are called a tensor characteristic of semi-reducibility. We study conditions of differentiating and their extensions for a tensor characteristic.

Namely, we proved that the vector of tensor characteristic of semi-reducibility lies in the core of the tensor of this characteristic. For some categories of specialized spaces (spaces of constant curvature, Ricci-symmetric spaces and conformal flat spaces) we found conditions, which limit these vector and tensor with a necessity. In order to get these results we applied a property of idempotence of a tensor of semi-reducibility characteristic. The condition of idempotence is important for semi-reducibility of spaces without limitations imposed on a sign of metric tensor.

Thus, the obtained results are interesting for the research on spaces with condition of idempotence. As it is known [18], the condition of idempotence can be weakened and replaces with the condition, that the matrix of tensor $b_{ij}$ should have simple elementary divisors and real roots. At least, in such a way, this condition is cited in the L. Eisenhart's book "Riemannian geometry" [2], but there is no part of the condition on real roots.

The particular attention is deserved by the research on positive conditions and conditions of global or topological type [18]. It will facilitate an effective research on geometric properties both generalized spaces and definite spaces of general theory of relativity.

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