Reversing orientation homeomorphisms of surfaces

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To the memory of Volodymyr Vasylyovych Sharko

Abstract. Let M be a connected compact orientable surface, $f : M \rightarrow \mathbb{R}$ be a Morse function, and $h : M \rightarrow M$ be a diffeomorphism which preserves f in the sense that $f \circ h = f$. We will show that if h leaves invariant each regular component of each level set of f and reverses its orientation, then h^2 is isotopic to the identity map of M via f -preserving isotopy.

This statement can be regarded as a foliated and a homotopy analogue of a well known observation that every reversing orientation orthogonal isomorphism of a plane has order 2, i.e. a mirror symmetry with respect to some line.

The obtained results hold in fact for a larger class of maps with isolated singularities from compact orientable surfaces to the real line and the circle.

Анотація. Нехай $f : M \rightarrow \mathbb{R}$ – функція Морса на зв'язній компактній орієнтовній поверхні M , і $h : M \rightarrow M$ – дифеоморфізм, що зберігає f , тобто $f \circ h = f$. В роботі показано, що коли h лишає інваріантною кожен регулярну компоненту кожної множини рівня f і змінює її орієнтацію, то його квадрат h^2 ізотопний до тотожного відображення id_M за допомогою ізоотопії, що також зберігає f .

Це твердження є одночасно пошаровим та гомотопічним аналогом добре відомого факту кожне ортогональне перетворення площини, яке змінює орієнтацію має порядок 2, і є дзеркальною симетрією відносно деякої прямої.

Отримані результати мають місце для більш широкого класу гладких відображень компактних поверхонь в пряму та коло які мають «однорідні» особливості.

Keywords: Diffeomorphism, Morse function, dihedral group

Ключові слова: Дифеоморфізм, відображення Морса, група дієдра

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1. INTRODUCTION

The present paper describes several *foliated* and *homotopy* variants of a “rigidity” property for reversing orientation linear motions of the plane claiming that *every such motion has order 2*. Though it is motivated by study of deformations of smooth functions on surfaces (and we prove the corresponding statements), the obtained results seem to have an independent interest.

Let $\mathbb{D}_n = \{r, s \mid r^n = s^2 = 1, rs = sr^{-1}\}$ be the *dihedral* group, i.e. group of symmetries of a right n -polygon. Then each “reversing orientation” element is written as $r^k s$ for some k and has order 2:

$$(r^k s)^2 = r^k s r^k s = r^k r^{-k} s s = 1.$$

More generally, let $SO^-(2) = \left\{ \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \mid t \in [0; 2\pi) \right\}$ be the *set* of reversing orientation orthogonal maps of $\mathbb{R}^2$, i.e. the adjacent class of $O(2)/SO(2)$ distinct from $SO(2)$. Then again one easily checks that each element of $SO^-(2)$ has order 2.

Another counterpart of this effect is that every motion of $\mathbb{R}$ which reverses orientation is given by the formula: $f(x) = a - x$ for some $a \in \mathbb{R}$, and therefore it has order 2, i.e. $f(f(x)) = x$.

Notice that such a property does not hold for “non-rigid motions”, like arbitrary homeomorphisms or diffeomorphisms of $\mathbb{R}$ or S^1 . For example, let $h : \mathbb{R} \rightarrow \mathbb{R}$ be given by $h(x) = -x^3$. It reverses orientation, but $h(h(x)) = -(-x^3)^3 = x^9$, and therefore it is not the identity.

Nevertheless, counterparts of the above rigidity effects for homeomorphisms can still be obtained on a “homotopy” level.

For instance, let $\mathcal{H}(S^1)$ be the group of all homeomorphisms of the circle S^1 and $\mathcal{H}^+(S^1)$, (resp. $\mathcal{H}^-(S^1)$), be the subgroup (resp. subset) consisting of homeomorphisms preserving (resp. reversing) orientation. Endow these spaces with compact open topologies. Notice that we have a natural inclusion $O(2) \subset \mathcal{H}(S^1)$, which *consists of two inclusions* $SO^-(2) \subset \mathcal{H}^-(S^1)$ and $SO(2) \subset \mathcal{H}^+(S^1)$ between the corresponding path components. It is well known ([3], see also Lemma 5.1 below) and is easy to show that $SO^-(2)$ (resp. $SO(2)$) is a strong deformation retract of $\mathcal{H}^-(S^1)$ (resp. $\mathcal{H}^+(S^1)$). This implies that the map $\text{sq} : \mathcal{H}^-(S^1) \rightarrow \mathcal{H}^+(S^1)$ defined by $\text{sq}(h) = h^2$ is null homotopic.

The aim of the present paper is to prove a parametric variant of the above “rigidity” statements for self-homeomorphisms of open subsets of topological products $X \times S^1$ preserving first coordinate (Theorem 6.2). That result will be applied to diffeomorphisms preserving a Morse function on an orientable surface and reversing certain regular components of some of its level-sets (Theorems 3.3, 3.5, and 3.6).

2. PRELIMINARIES

2.1. Action of the groups of diffeomorphisms. Let M be a compact surface and P be either the real line $\mathbb{R}$ or the circle S^1 and $\mathcal{D}(M)$ be the group of $\mathcal{C}^\infty$ diffeomorphisms of M . Then there is a natural right action of $\mathcal{D}(M)$ on the space of smooth maps $\mathcal{C}^\infty(M, P)$ defined by the following rule: $(h, f) \mapsto f \circ h$, where $h \in \mathcal{D}(M)$, $f \in \mathcal{C}^\infty(M, P)$. Let

$$\mathcal{S}(f) = \{h \in \mathcal{D}(M) \mid f \circ h = f\}, \quad \mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}(M)\},$$

be the *stabilizer* and the *orbit* of $f \in \mathcal{C}^\infty(M, P)$ with respect to the above action. It will be convenient to say that elements of $\mathcal{S}(f)$ *preserve* f . Endow the spaces $\mathcal{D}(M)$, $\mathcal{C}^\infty(M, P)$ with Whitney $\mathcal{C}^\infty$ -topologies, and their subspaces $\mathcal{S}(f)$, $\mathcal{O}(f)$ with induced ones. Denote by $\mathcal{S}_{\text{id}}(f)$ the identity path component of $\mathcal{S}(f)$ consisting of all $h \in \mathcal{S}(f)$ isotopic to id_M by some f -preserving isotopy.

For orientable M we denote by $\mathcal{D}^+(M)$ the group of its orientation preserving diffeomorphisms. Also for $f \in \mathcal{C}^\infty(M, P)$ we put

$$\mathcal{S}^+(f) = \mathcal{D}^+(M) \cap \mathcal{S}(f), \quad \mathcal{S}^-(f) = \mathcal{S}(f) \setminus \mathcal{S}^+(f),$$

so $\mathcal{S}^-(f)$ is another adjacent class of $\mathcal{S}(f)$ by $\mathcal{S}^+(f)$ distinct from $\mathcal{S}^+(f)$.

2.2. Homogeneous polynomials on $\mathbb{R}^2$ without multiple factors. It is well known and is easy to prove that every homogeneous polynomial $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a product of finitely many linear and irreducible over $\mathbb{R}$ quadratic factors:

$$f = \prod_{i=1}^l L_i \cdot \prod_{j=1}^q Q_j,$$

where $L_i(x, y) = a_i x + b_i y$ and $Q_j(x, y) = c_j x^2 + d_j xy + e_j y^2$.

Suppose $\deg f \geq 2$. Then the origin 0 is a unique critical point of f if and only if f has no multiple (non-proportional each other) linear factors. Structure of level sets of f near 0 is shown in Figure 2.1.

FIGURE 2.1. Topological structure of level sets of a homogeneous polynomial $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple factors

We will restrict ourselves with the following space of smooth maps.

Definition 2.2.1. Let $\mathcal{F}(M, P)$ be the set of $\mathcal{C}^\infty$ maps $f : M \rightarrow P$ satisfying the following conditions:

- (A1) The map f takes constant value at each connected component of ∂M and has no critical points in ∂M .
- (A2) For every critical point z of f there is a local presentation $f_z : \mathbb{R}^2 \rightarrow \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple factors.

A map $f \in \mathcal{C}^\infty(M, P)$ will be called *Morse* if f satisfies (A1) and all its critical points are non-degenerate. Notice that due to Morse lemma each Morse map f satisfies the condition (A2) with homogeneous polynomials $f_z = \pm x^2 \pm y^2$ for each critical point z . Denote by $\text{Morse}(M, P)$ the set of all Morse maps $M \rightarrow P$. Then we have the following inclusion:

$$\text{Morse}(M, P) \subset \mathcal{F}(M, P) \subset \mathcal{C}^\infty(M, P).$$

Then $\text{Morse}(M, P)$ is open and everywhere dense in the subset of smooth functions from $\mathcal{C}^\infty(M, P)$ satisfying the condition (A2). Hence $\mathcal{F}(M, P)$ consists of *even more that typical maps* $M \rightarrow P$.

2.3. Several constructions associated with $f \in \mathcal{F}(M, P)$. In what follows we will assume that $f \in \mathcal{F}(M, P)$. Let Σ_f be the set of critical points of f . Then condition (A2) implies that each $z \in \Sigma_f$ is isolated. In the case when $P = S^1$ one can also say about local extremes of f , and even about local minimums or maximums if we fix an orientation of S^1 .

A connected component K of a level-set $f^{-1}(c)$, $c \in P$, will be called a *leaf* (of f). We will call K *regular* if it contains no critical points. Otherwise, it will be called *critical*.

For $\varepsilon > 0$ let N_ε be the connected component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ containing K . Then N_ε will be called an *f -regular neighborhood of K* if ε is so small that $N_\varepsilon \setminus K$ contains no critical points of f and no boundary components of ∂M .

A submanifold $U \subset M$ will be called *f -adapted* if $U = \bigcup_{i=1}^a A_i$, where each A_i is either a regular leaf of f or an f -regular neighborhood of some (regular or critical) leaf of f .

2.3.1. Graph of f . Let Γ_f be the partition of M into leaves of f , and $p : M \rightarrow \Gamma_f$ be the natural map associating to each $x \in M$ the corresponding leaf of f containing x . Endow Γ_f with the quotient topology, so a subset (a collection of leaves) $U \subset \Gamma_f$ is open iff $p^{-1}(U)$ (i.e. their union) is open in M . It follows from axioms (A1) and (A2) that Γ_f has a natural structure of 1-dimensional CW-complex, whose 0-cells correspond to

boundary components of M and critical leaves of f . It is known as Reeb or Kronrod-Reeb or Lyapunov graph of f , see [1, 2, 4, 10]. We will call Γ_f simply the *graph* of f .

The following statement is evident:

Lemma 2.3.2. *For $f \in \mathcal{F}(M, P)$ the following conditions are equivalent:*

- (a) *every regular leaf of f in $\text{Int } M$ separates M ;*
- (b) *the graph Γ_f of f is a tree.*

For instance, those conditions hold if M is either of the surfaces: 2-disk, cylinder, 2-sphere, Möbius band, projective plane. $\square$

2.3.3. Singular foliation of f . Consider finer partition Ξ_f of M whose elements are of the following three types:

- (i) regular leaves of f ;
- (ii) critical points of f ;
- (iii) connected components of the sets $K \setminus \Sigma_f$, where K runs over all critical leaves of f (evidently, each such component is an open arc).

In other words, each critical leaf of f is additionally partitioned by critical points. We will call Ξ_f the *singular foliation* of f .

2.3.4. Hamiltonian like flows of $f \in \mathcal{F}(M, P)$. Suppose M is orientable. A smooth vector field F on M will be called *Hamiltonian-like* for f if the following conditions hold true.

- (a) $F(z) = 0$ if and only if z is a critical point of f .
- (b) $df(F) \equiv 0$ everywhere on M , i.e. f is constant along orbits of F .
- (c) Let z be a critical point of f . Then there exists a local representation of f at z in the form of a homogeneous polynomial $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple factors as in Axiom (A2), such that in the same coordinates (x, y) near the origin 0 in $\mathbb{R}^2$ we have that $F = -g'_y \frac{\partial}{\partial x} + g'_x \frac{\partial}{\partial y}$.

It follows from (a) and Axiom (A1) that the orbits of F precisely elements of the singular foliation Ξ_f .

By [7, Lemma 5.1] or [8, Lemma 16] every $f \in \mathcal{F}(M, P)$ admits a Hamiltonian-like vector field.

The following statement is a principal technical result established in a series of papers by the second author, see [9, Lemma 6.1(iv)] for details:

Lemma 2.3.5. *Let $\mathbf{F} : M \times \mathbb{R} \rightarrow M$ be a Hamiltonian like flow for f and $h \in \mathcal{S}(f)$. Then $h \in \mathcal{S}_{\text{id}}(f)$ if and only if there exists a C^∞ function $\alpha : M \rightarrow \mathbb{R}$ such that $h(x) = \mathbf{F}(x, \alpha(x))$ for all $x \in M$. Moreover, in this case the homotopy $H : M \times [0; 1] \rightarrow M$ given by $H(x, t) = \mathbf{F}(x, t\alpha(x))$ is an isotopy between $H_0 = \text{id}_M$ and $H_1 = h$ in $\mathcal{S}(f)$.*

3. MAIN RESULTS

Let M be a compact surface, $h : M \rightarrow M$ a homeomorphism, and $\gamma \subset M$ a submanifold of M . Then γ is *h -invariant*, whenever $h(\gamma) = \gamma$.

Moreover, suppose γ is connected and *orientable* h -invariant submanifold with $\dim \gamma \geq 1$. Then γ is *h^+ -invariant* (resp. *h^- -invariant*) whenever the restriction $h|_\gamma : \gamma \rightarrow \gamma$ preserves (resp. reverses) orientation of γ . If γ is a fixed point of h , then we assume that γ is mutually h^+ - and h^- -invariant.

We will be interesting in the structure of diffeomorphisms preserving $f \in \mathcal{F}(M, P)$ and reversing orientations of some regular leaves of f . The following easy lemma can be proved similarly to [7, Lemma 3.5].

Lemma 3.1. *Let M be a connected orientable surface and $h \in \mathcal{S}(f)$ be such that every regular leaf of f is h -invariant. Then every critical leaf of f is also h -invariant and the following conditions are equivalent:*

- (1) *some regular leaf of f is h^+ -invariant;*
- (2) *all regular leaves of f are h^+ -invariant;*
- (3) *h preserves orientation of M .*

Counterexample 3.2. Lemma 3.1 fails for non-orientable surfaces. Let M be a Möbius band, and $f : M \rightarrow \mathbb{R}$ be a Morse function having two critical points: one local extreme z and one saddle y . Let K be the critical leaf of f containing y , and D and E be the connected component of $M \setminus K$ containing z and ∂M respectively. Then it is easy to construct a diffeomorphism $h : M \rightarrow M$ (called *slice along central circle of Möbius band*) which is fixed on ∂M , and for which D is h^- -invariant. Then regular leaves of f in D are h^- -invariant, while regular leaves of f in E are h^+ -invariant.

Denote by $\Delta^-(f)$ the subset of $\mathcal{S}(f)$ consisting of diffeomorphisms h such that every regular leaf of f is h^- -invariant.

Let $h \in \Delta^-(f)$. Then by Lemma 3.1 every critical leaf K of f is h^- -invariant, however, h may interchange critical points of f in K and the leaves of Ξ_f contained in K (i.e. connected components of $K \setminus \Sigma_f$). Notice also that in general $\Delta^-(f)$ can be empty.

The following theorem can be regarded as a *homotopical* and *foliated* variant of the above rigidity property for diffeomorphisms preserving f .

Theorem 3.3. *Let M be a compact orientable surface, $f \in \mathcal{F}(M, P)$, and $h \in \Delta^-(f)$. Then $h^2 \in \mathcal{S}_{\text{id}}(f)$. In particular, for every $h \in \Delta^-(f)$ each leaf of Ξ_f is $(h^2, +)$ -invariant.*

Corollary 3.4. *Let $\text{or} : \pi_0 \mathcal{S}(f) \rightarrow \mathbb{Z}_2 = \{0, 1\}$ be the orientation homomorphism defined by $\text{or}(\mathcal{S}^+(f)) = 0$ and $\text{or}(\mathcal{S}^-(f)) = 1$. Then for each*

$h \in \Delta^-(f)$ the map $s : \mathbb{Z}_2 \rightarrow \pi_0 \mathcal{S}(f)$, defined by $s(0) = [\text{id}_M]$ and $s(1) = [h]$ is a homomorphism satisfying $\text{or} \circ s = \text{id}_{\mathbb{Z}_2}$. In other words, s is a section of or , whence $\pi_0 \mathcal{S}(f)$ is a certain semidirect product of $\pi_0 \mathcal{S}^+(f)$ and $\mathbb{Z}_2$.

Proof. One should just mention that Theorem 3.3 implies that $[h]^2 = [\text{id}_M]$ in $\pi_0 \mathcal{S}(f)$. $\square$

Now consider the situation, when not all regular leaves are h^- -invariant. To clarify the situation we first formulate a particular case of the general statement for maps on 2-disk and cylinder.

Theorem 3.5. *Let M be a compact orientable surface, $f \in \mathcal{F}(M, P)$, V be a regular leaf of f , and $h \in \mathcal{S}(f)$. Suppose every regular leaf of f in $\text{Int } M$ separates M (this holds e.g. when M is a 2-disk, cylinder or a 2-sphere, see Lemma 2.3.2), and that V is h^- -invariant. Then there exists $g \in \mathcal{S}(f)$ which coincide with h on some neighborhood of V and such that $g^2 \in \mathcal{S}_{\text{id}}(f)$.*

Emphasize that Theorem 3.5 *does not claim* that $g \in \Delta^-(f)$, though g reverses orientation of V and its square belong to $\mathcal{S}_{\text{id}}(f)$.

General result. Theorems 3.3 and 3.5 are consequences of the following Theorem 3.6 below. Let M be a compact (not necessarily orientable) surface, $f \in \mathcal{F}(M, P)$, and $h \in \mathcal{S}(f)$. Let also

- A be the union of all h^- -invariant regular leaves of f ,
- $K_1, \dots, K_k$ be all the critical leaves of f such that $\overline{A} \cap K_i \neq \emptyset$;
- for $i = 1, \dots, k$, let R_{K_i} be an f -regular neighborhood of K_i chosen so that $R_{K_i} \cap R_{K_j} = \emptyset$ for $i \neq j$ and

$$Z := A \bigcup \left(\bigcup_{i=1}^k R_{K_i} \right).$$

Evidently, Z is an f -adapted subsurface of M and each of its connected components intersects A .

Lemma 3.5.1. *Suppose Z is orientable. Let also γ be a boundary component of Z . Consider the following conditions:*

- (1) $\gamma \subset \text{Int } M$;
- (2) $\gamma = \partial U \cap \partial Z$ for some connected component U of $\overline{M \setminus Z}$;
- (3) $h(\gamma) \cap \gamma = \emptyset$;
- (4) $h(\gamma) \neq \gamma$.

Then $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$.

Proof. $(1) \Leftrightarrow (2)$ is evident, and $(3) \Leftrightarrow (4)$ follows from the observation that γ is a regular leaf of h and h permutes leaves of f .

(2) $\Rightarrow$ (4) Suppose $h(\gamma) = \gamma$. Let Z' be a connected component of Z containing γ . Then by the construction Z' must intersect A , whence h changes orientation of some regular leaves in Z' . Since Z' is also orientable, we get from Lemma 3.1 that h also changes orientation of γ . This means that $\gamma \subset A$, and therefore there exists an open neighborhood $W \subset A$ of γ consisting of regular leaves of f . In particular, the regular leaves in $W \cap \text{Int } U$ must be contained in A which contradicts to the assumption that $\text{Int } U \subset M \setminus Z \subset M \setminus A$. $\square$

Thus h interchanges boundary components of Z belonging to the interior of M . We will introduce the following property on Z :

(B) *every connected component of $\partial Z \cap \text{Int } M$ separates M .*

This condition means that there is a bijection between boundary components of $\partial Z \cap \text{Int } M$ and connected components of $M \setminus Z$, whence by Lemma 3.5.1 there will be no h -invariant connected components of $M \setminus Z$.

Theorem 3.6. *If Z is non-empty, orientable and has property (B), then there exists $g \in \mathcal{S}(f)$ such that $g = h$ on Z and $g^2 \in \mathcal{S}_{\text{id}}(f)$.*

Proof of Theorem 3.3. Suppose M is orientable and $h \in \Delta^-(f)$. Then in the notation of Theorem 3.6, $Z = M$, and by that theorem there exists $g \in \mathcal{S}(f)$ such that $g = h$ on Z and $g^2 \in \mathcal{S}_{\text{id}}(f)$. This means that $h = g$ and $h^2 \in \mathcal{S}_{\text{id}}(f)$. $\square$

Proof of Theorem 3.5. The assumption that every regular leaf of f in $\text{Int } M$ separates M implies condition (B). $\square$

3.7. Structure of the paper. In Section 4 we discuss a notion of a shift map along orbits of a flow which was studied in a series of papers by the second author, and extend several results to continuous flows. Section 5 devoted to reversing orientation families of homeomorphisms of the circle. In Section 6 we study flows without fixed point, and in Section 7 recall several results about passing from a flow on the plane to the flow written in polar coordinates. Section 8 introduces a certain subsurfaces of a surface M associated with a map $f \in \mathcal{F}(M, P)$ and called *chipped cylinders*. We prove Theorem 8.3 describing behavior of diffeomorphisms reversing regular leaves of f contained in those chipped cylinders. In Section 9 we prove Lemma 9.1 allowing to change f -preserving diffeomorphisms so that its finite power will be isotopic to the identity by f -preserving isotopy. Finally, in sections 9 and 10 we prove Theorem 3.6.

4. SHIFTS ALONG ORBITS OF FLOWS

In this section we extend several results obtained in [5, 9] for smooth flows to a continuous situation. Let X be a topological space.

Definition 4.1. A continuous map $\mathbf{F} : X \times \mathbb{R} \rightarrow X$ is a (*global*) *flow on* X , if $\mathbf{F}_0 = \text{id}_X$ and $\mathbf{F}_s \circ \mathbf{F}_t = \mathbf{F}_{s+t}$ for all $s, t \in \mathbb{R}$, where $\mathbf{F}_s : X \rightarrow X$ is given by $\mathbf{F}_s(x) = \mathbf{F}(x, s)$. For $x \in X$ the subset $\mathbf{F}(x \times \mathbb{R}) \subset X$ is called the *orbit* of x .

It is well known that a $\mathcal{C}^r$, $1 \leq r \leq \infty$, vector field F of a smooth compact manifold X tangent to ∂X always generates a flow $\mathbf{F}$.

Assume further that $\mathbf{F}$ is a flow on a topological space X . Let also $V \subset X$ be a subset. Say that a continuous map $h : V \rightarrow X$ *preserves orbits of $\mathbf{F}$ on V* if $h(\gamma \cap V) \subset \gamma$ for every orbit γ of $\mathbf{F}$. The latter means that for each $x \in V$ there exists a number $\alpha_x \in \mathbb{R}$ such that $h(x) = \mathbf{F}(x, \alpha_x)$. Notice that in general α_x is not unique and does not continuously depend on x .

Conversely, let $\alpha : V \rightarrow \mathbb{R}$ be a continuous function such that its graph $\Gamma_\alpha = \{(x, \alpha(x)) \mid x \in V\}$ is contained in W . For a global flow any continuous $\alpha : V \rightarrow \mathbb{R}$ satisfies that condition. Then one can define the following map $\mathbf{F}_\alpha : V \rightarrow X$ by

$$\mathbf{F}_\alpha(z) := \mathbf{F}(z, \alpha(z)) = \mathbf{F}_{\alpha(z)}(z).$$

We will call $\mathbf{F}_\alpha$ a *shift along orbits of $\mathbf{F}$ by the function α* , while α will be called a *shift function* for $\mathbf{F}_\alpha$.

Notice that $\mathbf{F}_\alpha$ preserves orbits of $\mathbf{F}$ on V , and in general is not a homeomorphism.

Definition 4.2. Let x be a non-fixed point of $\mathbf{F}$, and Y be a topological space. Let also U be an open neighborhood of x , and $\phi = (\zeta, \rho) : U \rightarrow Y \times \mathbb{R}$ be an open embedding. Then the pair (ϕ, U) will be called a *flow-box chart* at x , if there exist an $\varepsilon > 0$ and an open neighborhood V of x in X such that

$$\phi \circ \mathbf{F}(z, t) = (\zeta(z), \rho(z) + t) \quad (4.1)$$

for all $(z, t) \in V \times (-\varepsilon; \varepsilon)$.

In other words, $\mathbf{F}$ is *locally conjugated* to the flow $\mathbf{G} : (Y \times \mathbb{R}) \times \mathbb{R} \rightarrow Y \times \mathbb{R}$ defined by $\mathbf{G}(y, s, t) = (y, s + t)$, since the identity (4.1) can be written as

$$\phi \circ \mathbf{F}(z, t) = (\zeta(z), \rho(z) + t) = \mathbf{G}(\zeta(z), \rho(z), t) = \mathbf{G}(\phi(z), t),$$

i.e.

$$\phi \circ \mathbf{F}_t(z) = \mathbf{G}_t \circ \phi(z). \quad (4.2)$$

It is well known that each $\mathcal{C}^2$ flow (generated by some $\mathcal{C}^1$ vector field) on a smooth manifold admits flow-box charts at each non-fixed point.

Lemma 4.3. *If (ϕ, U) is a flow box chart at a non-fixed point $x \in X$ of $\mathbf{F}$, then for any $\tau \in \mathbb{R}$ the pair $(\phi \circ \mathbf{F}_{-\tau}, \mathbf{F}_\tau(U))$ is a flow box chart at the point $y = \mathbf{F}(x, \tau)$.*

Proof. Denote $U' = \mathbf{F}_\tau(U)$ and $\phi' = \phi \circ \mathbf{F}_{-\tau}$. Let also ε and V be as in Definition 4.1, and $V' = \mathbf{F}_\tau(V)$ be a neighborhood of y . Then for any $(z, t) \in V' \times (-\varepsilon; \varepsilon)$ we have that

$$\begin{aligned} \phi' \circ \mathbf{F}_t(z) &= (\phi \circ \mathbf{F}_{-\tau}) \circ \mathbf{F}_t(z) = (\phi \circ \mathbf{F}_{t-\tau})(z) = \\ &= (\phi \circ \mathbf{F}_t(\mathbf{F}_{-\tau}(z))) \stackrel{(4.1)}{=} \mathbf{G}_t \circ \phi(\mathbf{F}_{-\tau}(z)) = \mathbf{G}_t \circ \phi'(z). \quad \square \end{aligned}$$

The following lemma shows that for flows admitting flow box charts (e.g. for smooth flows) every orbit preserving map admits a shift function near each non-fixed point. Moreover, such a function is locally determined by its value at that point.

Lemma 4.4. (cf. [9, Lemma 6.1(i)]). *Suppose a flow $\mathbf{F} : X \times \mathbb{R} \rightarrow X$ has a flow-box (ϕ, U) at some non-fixed point x . Let also $h : U \rightarrow X$ be a continuous map preserving orbits of $\mathbf{F}$ and such that $h(x) = \mathbf{F}(x, \tau)$ for some $\tau \in \mathbb{R}$. Then there exists an open neighborhood $V \subset U$ of x and a unique continuous function $\alpha : V \rightarrow \mathbb{R}$ such that*

- (1) $\alpha(x) = \tau$;
- (2) $h(z) = \mathbf{F}(z, \alpha(z))$ for all $z \in V$.

If in addition X is a manifold of class $\mathcal{C}^r$, ($1 \leq r \leq \infty$), $\mathbf{F}$ and h are $\mathcal{C}^r$, and ϕ is a $\mathcal{C}^r$ embedding, then α is $\mathcal{C}^r$ as well.

Proof. The proof almost literally repeats the arguments of [9, Lemma 6.2] proved for smooth flows and based on existence of flow box charts. For completeness we present a short proof for continuous situation.

1) First suppose $\tau = 0$, so $h(x) = x$. Let V and ε be the same as in Definition 4.2. Decreasing V one can also assume that $V \subset U \cap h^{-1}(U)$, so in particular $h(V) \subset U$. Denote $\hat{V} := \phi(V)$ and $\hat{U} := \phi(U)$. Then these sets are open, and we have a well-defined map $\hat{h} = \phi \circ h \circ \phi^{-1} : \hat{V} \rightarrow \hat{U}$ which preserves orbits of $\mathbf{G}$ due to (4.2). This means that $\hat{h}(y, t) = (y, \eta(y, t))$ for some continuous function $Y \times \mathbb{R} \supset \hat{V} \xrightarrow{\eta} \mathbb{R}$. Define another continuous function $\alpha' : \hat{V} \rightarrow \mathbb{R}$ by $\alpha'(y, t) = \eta(y, t) - t$. Then

$$\hat{h}(y, t) = (y, t + \alpha'(y, t)) = \mathbf{G}(y, t, \alpha'(y, t)) = \mathbf{G}_{\alpha'}(y, t),$$

that is $\phi \circ h \circ \phi^{-1} = \hat{h} = \mathbf{G}_{\alpha'}$, whence for each $z \in V$ we have that

$$h(z) = \phi^{-1} \circ \mathbf{G}_{\alpha'} \circ \phi(z) = \phi^{-1} \circ \mathbf{G}_{\alpha' \circ \phi(z)} \circ \phi(z) = \mathbf{F}_{\alpha' \circ \phi(z)}(z) = \mathbf{F}_{\alpha' \circ \phi}(z).$$

Thus one can put $\alpha = \alpha' \circ \phi : V \rightarrow \mathbb{R}$. It follows that α is continuous. Moreover, let $\hat{y}, \hat{t} = \phi(x)$. Since $h(x) = x$, we get that $\tilde{h}(\hat{y}, \hat{t}) = \hat{y}, \hat{t}$, whence $\eta(\hat{y}, \hat{t}) = \hat{t}$, whence $\alpha'(\hat{y}, \hat{t}) = 0$, and thus

$$\alpha(x) = \alpha' \circ \phi(x) = \alpha'(\hat{y}, \hat{t}) = 0.$$

2) If $\tau \neq 0$, then consider the map $\tilde{h} : U \rightarrow X$ given by $\tilde{h} = \mathbf{F}_{-\tau} \circ h$. Then $\tilde{h}(x) = \mathbf{F}_{-\tau} \circ h(x) = \mathbf{F}(\mathbf{F}(x, \tau), -\tau) = x$, whence by 1) $\tilde{h} = \mathbf{F}_{\hat{\alpha}}$ for a unique continuous function $\hat{\alpha} : V \rightarrow \mathbb{R}$ such that $\hat{\alpha}(x) = 0$. Put $\alpha = \hat{\alpha} + \tau$. Then $\alpha(x) = \tau$ and

$$h(z) = \mathbf{F}_{\tau} \circ \tilde{h}(z) = \mathbf{F}(\mathbf{F}(z, \hat{\alpha}(z)), \tau) = \mathbf{F}(z, \hat{\alpha}(z) + \tau) = \mathbf{F}(z, \alpha(z)). \quad \square$$

The formulas for α imply that if X , $\mathbf{F}$, ϕ , and h are $\mathcal{C}^r$, ($1 \leq r \leq \infty$), then α is $\mathcal{C}^r$ as well.

Corollary 4.5. *Suppose $U \subset X$ is an open **connected** subset such that every $x \in U$ is non-fixed and admits a flow-box. Let also $\alpha, \alpha' : U \rightarrow \mathbb{R}$ be two continuous functions such that $\mathbf{F}_{\alpha} = \mathbf{F}_{\alpha'}$ on U . If $\alpha(x) = \alpha'(x)$ at some $x \in U$ (this holds e.g. if $\mathbf{F}$ has at least one non-periodic point in U), then $\alpha = \alpha'$ on U .*

Proof. The set $A = \{\alpha(y) = \alpha'(y) \mid y \in U\}$ is closed and by assumption non-empty (contains x). Moreover, by Lemma 4.4 this set is open, whence $A = U$. $\square$

Corollary 4.6. *Let $\mathbf{F} : X \times \mathbb{R} \rightarrow X$ be a continuous flow, $U \subset X$ be an open subset such that every point $x \in U$ is non-fixed and non-periodic and admits a flow box chart, and $h : U \rightarrow X$ be an orbit preserving map. Then there exists a unique continuous function $\alpha : U \rightarrow \mathbb{R}$ such that $h(x) = \mathbf{F}(x, \alpha(x))$ for all $x \in U$.*

If in addition X is a manifold of class $\mathcal{C}^r$, ($0 \leq r \leq \infty$), $\mathbf{F}$ is $\mathcal{C}^r$ and admits $\mathcal{C}^r$ flow box charts, and h is $\mathcal{C}^r$, then α is $\mathcal{C}^r$ as well.

Proof. Since every point $x \in U$ is non-fixed and non-periodic, there exists a unique number $\alpha(x)$ such that $h(x) = \mathbf{F}(x, \alpha(x))$. Moreover, since $\mathbf{F}$ admits flow box chart at x , it follows from Lemma 4.4 that the correspondence $x \mapsto \alpha(x)$ is a continuous function $\alpha : U \rightarrow \mathbb{R}$, which is also $\mathcal{C}^r$ under the corresponding smoothness assumptions on X , $\mathbf{F}$, ϕ , and h . $\square$

The following statement extends some results established in [9] for smooth flows to continuous flows having flow box charts at each non-fixed point on arbitrary topological spaces.

Lemma 4.7. (c.f. [9, Lemmas 6.1-6.3]) *Let $\mathbf{F} : X \times \mathbb{R} \rightarrow X$ be a flow, $p : \tilde{X} \rightarrow X$ a covering map, and $\xi : \tilde{X} \rightarrow \tilde{X}$ a covering transformation, i.e. a homeomorphism such that $p \circ \xi = p$. Then the following statements hold.*

(1) $\mathbf{F}$ lifts to a unique flow $\tilde{\mathbf{F}} : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X}$ such that $p \circ \tilde{\mathbf{F}}_t = \mathbf{F}_t \circ p$ for all $t \in \mathbb{R}$.

(2) For each continuous function $\alpha : X \rightarrow \mathbb{R}$ the map $\tilde{\mathbf{F}}_{\alpha \circ p} : \tilde{X} \rightarrow \mathbb{R}$ is a lifting of $\mathbf{F}_\alpha$, that is

(3) $\tilde{\mathbf{F}}$ commutes with ξ in the sense that $\tilde{\mathbf{F}}_t \circ \xi = \xi \circ \tilde{\mathbf{F}}_t$ for all $t \in \mathbb{R}$. More generally, for any function $\beta : \tilde{X} \rightarrow \mathbb{R}$ we have $\tilde{\mathbf{F}}_\beta \circ \xi = \xi \circ \tilde{\mathbf{F}}_{\beta \circ \xi}$.

(4) For a continuous function $\beta : \tilde{X} \rightarrow \mathbb{R}$ and a point $z \in \tilde{X}$ consider the following (“global” and “point”) conditions:

$$\begin{array}{ll} \text{(g1)} \quad \beta = \beta \circ \xi; & \text{(p1)} \quad \beta(z) = \beta \circ \xi(z); \\ \text{(g2)} \quad \tilde{\mathbf{F}}_\beta = \tilde{\mathbf{F}}_{\beta \circ \xi}; & \text{(p2)} \quad \tilde{\mathbf{F}}_\beta(z) = \tilde{\mathbf{F}}_{\beta \circ \xi}(z); \\ \text{(g3)} \quad \tilde{\mathbf{F}}_\beta \circ \xi = \xi \circ \tilde{\mathbf{F}}_\beta; & \text{(p3)} \quad \tilde{\mathbf{F}}_\beta \circ \xi(z) = \xi \circ \tilde{\mathbf{F}}_\beta(z). \end{array}$$

Then we have the following diagram of implications:

$$\begin{array}{ccccc} \text{(g1)} & \implies & \text{(g2)} & \overset{(2)}{\iff} & \text{(g3)} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{(p1)} & \implies & \text{(p2)} & \overset{(2)}{\iff} & \text{(p3)} \end{array} \tag{4.3}$$

- (i) If z is non-fixed and non-periodic point for $\tilde{\mathbf{F}}$, then $(\text{p2}) \Rightarrow (\text{p1})$.
- (ii) If $\tilde{X}$ is path connected and $\tilde{\mathbf{F}}_\beta$ is a lifting of some continuous map $h : X \rightarrow X$, that is $h \circ p = p \circ \tilde{\mathbf{F}}_\beta$, then $(\text{p3}) \Rightarrow (\text{g3})$, whence all conditions in the right square of (4.3) are equivalent.
- (iii) If $\tilde{X}$ is path connected, and $\tilde{\mathbf{F}}$ has no fixed points and admits flow box charts at each point of $\tilde{X}$, then $(\text{g2}) \& (\text{p1}) \Rightarrow (\text{g1})$.

Proof. (1) Consider the following homotopy (with “open ends”)

$$\mathbf{G} = \mathbf{F} \circ (p \times \text{id}_{\mathbb{R}}) : \tilde{X} \times \mathbb{R} \rightarrow X, \quad \mathbf{G}(z, t) = \mathbf{F}(p(z), t),$$

and let $\tilde{\mathbf{F}} : \tilde{X} \times 0 \rightarrow \tilde{X}$ be given by $\tilde{\mathbf{F}}(z, 0) = z$. Then $\tilde{\mathbf{F}}$ is a lifting of $\mathbf{G}|_{\tilde{X} \times 0}$, that is $p \circ \tilde{\mathbf{F}}(z, 0) = p(z) = \mathbf{F}(p(z), 0) = \mathbf{G}(z, 0)$. Hence $\tilde{\mathbf{F}}$ extends to a unique lifting $\tilde{\mathbf{F}} : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X}$ of $\mathbf{G}$. One easily checks that this lifting is a flow on $\tilde{X}$.

(2) Let $z \in \tilde{X}$ and $t = \alpha \circ p(z)$. Then

$$p \circ \tilde{\mathbf{F}}_{\alpha \circ p}(z) = p \circ \tilde{\mathbf{F}}_t(z) \stackrel{(1)}{=} \mathbf{F}_t \circ p(z) = \mathbf{F}_{\alpha \circ p(z)} \circ p(z) = \mathbf{F}_\alpha \circ p(z).$$

(3) Notice that the map $\tilde{\mathbf{F}}' : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X}$ defined by $\tilde{\mathbf{F}}'_t = \xi \circ \tilde{\mathbf{F}}_t \circ \xi^{-1}$ is also a flow on $\tilde{X}$. Moreover,

$$p \circ \tilde{\mathbf{F}}'_t = p \circ \xi \circ \tilde{\mathbf{F}}'_t \circ \xi^{-1} = p \circ \tilde{\mathbf{F}}_t \circ \xi^{-1} = \mathbf{F}_t \circ p \circ \xi^{-1} = \mathbf{F}_t \circ p.$$

Thus $\tilde{\mathbf{F}}'$ and $\tilde{\mathbf{F}}$ are two liftings of $\mathbf{F}$ which coincide at $t = 0$, and therefore $\tilde{\mathbf{F}}' = \tilde{\mathbf{F}}$ by uniqueness of liftings. Therefore for any continuous function $\tilde{\alpha} : \tilde{X} \rightarrow \mathbb{R}$ and $z \in \tilde{X}$ we have, (cf. [9, Eq. (6.8)])

$$\begin{aligned} \tilde{\mathbf{F}}_{\tilde{\alpha}} \circ \xi(z) &= \tilde{\mathbf{F}}(\xi(z), \tilde{\alpha}(\xi(z))) = \tilde{\mathbf{F}}_{\tilde{\alpha}(\xi(z))} \circ \xi(z) \\ &= \xi \circ \tilde{\mathbf{F}}_{\tilde{\alpha} \circ \xi}(z) = \xi \circ \tilde{\mathbf{F}}_{\tilde{\alpha} \circ \xi}(z). \end{aligned}$$

(4) The implication in Diagram (4.3) are trivial. Assume that $\tilde{X}$ (and therefore X) are path connected.

(i) If z is non-fixed and non-periodic, then $\tilde{\mathbf{F}}(z, a) = \tilde{\mathbf{F}}(z, b)$ implies that $a = b$ for any a, b . In particular, this hold for $a = \beta(z)$ and $b = \beta \circ \xi(z)$.

(ii) Suppose that $\tilde{\mathbf{F}}_\beta$ is a lifting of some continuous map $h : X \rightarrow X$. Then $p \circ \tilde{\mathbf{F}}_\beta \circ \xi = h \circ p \circ \xi = h \circ p$ and $p \circ \xi \circ \tilde{\mathbf{F}}_\beta = p \circ \tilde{\mathbf{F}}_\beta = h \circ p$, i.e. both $\tilde{\mathbf{F}}_\beta \circ \xi$ and $\xi \circ \tilde{\mathbf{F}}_\beta$ are liftings of h .

Now if (p3) holds, i.e. $\tilde{\mathbf{F}}_\beta \circ \xi(z) = \xi \circ \tilde{\mathbf{F}}_\beta(z)$ at some point $z \in \tilde{X}$, then these liftings must coincide on all of $\tilde{X}$, which means condition (g3).

(iii) Suppose $\tilde{\mathbf{F}}$ has no fixed points and conditions (g2) and (p1) hold, that is $\tilde{\mathbf{F}}(y, \beta(y)) = \tilde{\mathbf{F}}(y, \beta \circ \xi(y))$ for all $y \in \tilde{X}$ and $\beta(z) = \beta \circ \xi(z)$ for some $z \in \tilde{X}$. Notice that the set $A = \{y \in \tilde{X} \mid \beta(y) = \beta \circ \xi(y)\}$ is close. Moreover, by the local uniqueness of shift-functions (Corollary 4.5) A is also open. Since $\tilde{X}$ is connected, A is either $\emptyset$ or $\tilde{X}$. But $z \in A$, whence $A = \tilde{X}$, i.e. condition (g1) holds. $\square$

Let $p : \tilde{X} \rightarrow X$ be a *regular* covering map with path connected $\tilde{X}$ and X , G be the group of covering transformation, $\mathbf{F} : X \times \mathbb{R} \rightarrow X$ be a flow on X , and $\tilde{\mathbf{F}} : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X}$ be its lifting as in Lemma 4.7(1).

Corollary 4.8. (c.f. [9, Lemma 6.3]) *Suppose all orbits of $\tilde{\mathbf{F}}$ are non-fixed and non-closed and $\tilde{\mathbf{F}}$ has flow box charts at all points of $\tilde{X}$. Let also $h : X \rightarrow X$ be a continuous map admitting a lifting $\tilde{h} : \tilde{X} \rightarrow \tilde{X}$, i.e. $p \circ \tilde{h} = h \circ p$, such that $\tilde{h}$ leaves invariant each orbit γ of $\tilde{\mathbf{F}}$ and commutes with each $\xi \in G$. Then there exist a unique continuous function $\alpha : X \rightarrow \mathbb{R}$ such that $h = \mathbf{F}_\alpha$.*

Again if $\tilde{X}$, X , p , h , $\mathbf{F}$ and its flow box charts are C^r , ($1 \leq r \leq \infty$), then α is also C^r .

Proof. Due to assumptions on $\tilde{\mathbf{F}}$ we get from Corollary 4.6 that there exists a unique continuous function $\beta : \tilde{X} \rightarrow \mathbb{R}$ such that $\tilde{h} = \tilde{\mathbf{F}}_\beta$. Let

$\xi \in G$. Since $\tilde{h}$ commutes with ξ , i.e. condition (g3) of Lemma 4.7 holds, we obtain the following implications:

$$(g3) \iff (g2) \implies (p2) \xRightarrow{(i)} (p1) \xRightarrow{(iii)} (g1).$$

meaning that $\beta \circ \xi = \beta$. Thus β is invariant with respect to all $\xi \in G$, and therefore it induces a unique function $\alpha : X \rightarrow \mathbb{R}$ such that $\beta = \alpha \circ p$. Since p is a local homeomorphism, it follows that α is continuous. Moreover, by Lemma 4.7(2), $\tilde{h} = \tilde{\mathbf{F}}_\beta = \tilde{\mathbf{F}}_{\alpha \circ p}$ is a lifting of $\mathbf{F}_\alpha$. But $\tilde{h}$ is also a lifting of h , whence $h = \mathbf{F}_\alpha$.

Statements about smoothness of α follows from the corresponding smoothness parts of used lemmas. We leave the details for the reader. $\square$

5. SELF MAPS OF THE CIRCLE

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle in the complex plane, $p: \mathbb{R} \rightarrow S^1$ be the universal covering map defined by $p(s) = e^{2\pi i s}$ and $\xi(s) = s + 1$ be a diffeomorphism of $\mathbb{R}$ generating the group of covering slices $\mathbb{Z}$.

Denote by $C_k(S^1, S^1)$, $k \in \mathbb{Z}$, the set of all continuous maps $h : S^1 \rightarrow S^1$ of degree k , i.e. maps homotopic to the map $z \mapsto z^k$. Then $\{C_k(S^1, S^1)\}_{k \in \mathbb{Z}}$ is a collection of all path components of $\mathcal{C}(S^1, S^1)$ with respect to the compact open topology.

For a map $h : X \rightarrow X$ it will be convenient to denote the composition $\underbrace{h \circ \dots \circ h}_n$ by h^n for $n \in \mathbb{N}$. A point $x \in X$ is *fixed* for h , whenever $h(x) = x$.

The following lemma collects simple properties of continuous maps of the circle which are usually referred as exercises. We present

Lemma 5.1. 1) Let $h : S^1 \rightarrow S^1$ be a continuous map and $\tilde{h}_0 : \mathbb{R} \rightarrow \mathbb{R}$ be any lifting of h with respect to p , i.e. a continuous map making commutative the following diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\tilde{h}_0} & \mathbb{R} \\ p \downarrow & & \downarrow p \\ S^1 & \xrightarrow{h} & S^1 \end{array}$$

that is $p \circ \tilde{h}_0 = h \circ p$, or $e^{2\pi i \tilde{h}_0(s)} = h(e^{2\pi i s})$ for $s \in \mathbb{R}$. For $a \in \mathbb{Z}$ define the map $\tilde{h}_a : \mathbb{R} \rightarrow \mathbb{R}$ by $\tilde{h}_a = \xi^a \circ \tilde{h}_0$, that is $\tilde{h}_a(s) = \tilde{h}_0(s) + a$. Then the following conditions are equivalent:

- (a) $h \in C_k(S^1, S^1)$;
- (b) $\tilde{h}_0 \circ \xi = \xi^k \circ \tilde{h}_0 = \tilde{h}_k$, that is $\tilde{h}(s + 1) = \tilde{h}(s) + k$ for all $s \in \mathbb{R}$.

2) Define the map $\phi_k : S^1 \rightarrow C_k(S^1, S^1)$, $\phi_k(w)(z) = wz^k$, $w, z \in S^1$, is a homotopy equivalence. Moreover, the inclusions $\phi_1 : SO(2) \subset \mathcal{H}^+(S^1)$ and $\phi_{-1} : SO^-(2) \subset \mathcal{H}^-(S^1)$ are homotopy equivalences as well.

3) Let $h \in C_k(S^1, S^1)$. Then

- (i) h has at least $|k - 1|$ fixed points;
- (ii) $\{\tilde{h}_{ak}\}_{a \in \mathbb{Z}}$ is the collection of all possible liftings of h ;
- (iii) $\tilde{h}_a \circ \tilde{h}_b = \xi^{a+kb} \circ h^2$ for any $a, b \in \mathbb{Z}$;
- (iv) if $k = -1$, then $\tilde{h}_a^2 = \tilde{h}_0^2$ for all $a \in \mathbb{Z}$, in other words, for any lifting $\tilde{h}_a$ of h_0 its square $\tilde{h}_a^2$ does not depend on a . Moreover, if A is the set of fixed points of h^2 , then $p^{-1}(A)$ is the set of fixed points of $\tilde{h}_0^2$.

Proof. Statement 1) is easy.

2) For $k = \pm 1$ this statement was initially proved by H. Kneser, see last paragraph on page 367 of [3]. Notice that

$$\phi_k(e^{2\pi is})(e^{2\pi ix}) = e^{2\pi i(s+kx)}.$$

for all $s, x \in \mathbb{R}$. In particular, ϕ_1 is the inclusion $SO(2) \subset \mathcal{H}^+(S^1)$, and ϕ_{-1} is the inclusion $SO^-(2) \subset \mathcal{H}^-(S^1)$. We will show that the image of ϕ_k is a strong deformation retract of $C_k(S^1, S^1)$.

Let $h \in C_k(S^1, S^1)$. Take any lifting $\tilde{h} : \mathbb{R} \rightarrow \mathbb{R}$ of h and define the following homotopy

$$H_{\tilde{h}} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}, \quad H_{\tilde{h}}(x, t) = (1 - t)\tilde{h}(x) + t(\tilde{h}(0) + kx).$$

Then it is easy to see that $H_{\tilde{h}}(x + 1, t) = H_{\tilde{h}}(x, t) + k$ for all $t, x \in \mathbb{R}$, and therefore it induces a certain homotopy $H_{\tilde{h}} : S^1 \times [0, 1] \rightarrow S^1$ between h and the map $\phi_k(h(1)) : z \mapsto h(1) \cdot z^k$.

Notice that if $\tilde{h}' = \tilde{h} + k$ is another lifting of h , then the analogous homotopy $H_{\tilde{h}'}(x, t) = (1 - t)(\tilde{h}'(x) + k) + t(\tilde{h}'(0) + kx)$ has the property that $H_{\tilde{h}'} = H_{\tilde{h}} + k$, whence $H_{\tilde{h}'}$ induces the same homotopy, i.e. $H_{\tilde{h}'} = H_{\tilde{h}}$. Hence $H_{\tilde{h}}$ depends only on H , and we will redenote it by H_h .

One easily checks that the map $H : C_k(S^1, S^1) \times [0, 1] \rightarrow C_k(S^1, S^1)$, $H(h, t)(\cdot) = H_h(\cdot, t)$, is continuous with respect to the compact open topology on $C_k(S^1, S^1)$. Moreover, if h belongs to the image of ϕ_k , then H_h is a constant homotopy at h . Hence H is the desired strong deformation retraction of $C_k(S^1, S^1)$ onto the image of ϕ_k .

Finally, if h is a homomorphism, then so is any of its liftings $\tilde{h}$, and the corresponding homotopies $H_{\tilde{h}}$ and H_h also consist of homeomorphisms. This implies that the inclusions $\phi_{\pm 1} : O(2) \subset \mathcal{H}(S^1)$ is a strong deformation retraction as well.

3) Statement (i) is a consequence of intermediate value theorem, and (ii) follows from (b).

$$(iii) \quad \tilde{h}_a \circ \tilde{h}_b = \xi^a \circ \tilde{h}_0 \circ \xi^b \circ \tilde{h}_0 = \xi^a \circ \xi^{kb} \circ \tilde{h}_0^2 = \xi^{a+kb} \circ \tilde{h}_0^2.$$

$$(iv) \quad \text{If } k = -1 \text{ then, by (iii), } \tilde{h}_a \circ \tilde{h}_a = \xi^{a-a} \circ \tilde{h}_0^2 = \tilde{h}_0^2.$$

Hence one can put $g := \tilde{h}_0^2 = \tilde{h}_a^2$ and this map does not depend on $a \in \mathbb{Z}$. Let also $\tilde{A}$ be the set of fixed points of g . We have to show that $\tilde{A} = p^{-1}(A)$.

Let $s \in \tilde{A}$, and $z = p(s)$. Then $s = g^2(s) = \tilde{h}_a^2(s)$ implies that

$$z = p(s) = p \circ \tilde{h}_a^2(s) = h^2 \circ p(s) = h^2(z),$$

so $z \in A$, that is $p(\tilde{A}) \subset A$ and thus $\tilde{A} \subset p^{-1}(A)$.

Conversely, let $z \in A$ and $s \in \mathbb{R}$ be such that $z = p(s)$. Then there exists a unique lifting $\tilde{h}_a$ of h such that $\tilde{h}_a(s) = s$. But then $g(s) = \tilde{h}_a^2(s) = s$, whence $s \in \tilde{A}$. $\square$

The following example shows that the effect described in the statement (iv) of Lemma 5.1 includes the *rigidity* property of reflections of the circle mentioned in the introduction.

Example 5.2. Let $h(z) = \overline{ze^{-2\pi\phi}}e^{2\pi\phi} = \bar{z}e^{2\phi}$ be a reflection of the complex plane with respect to the line passing through the origin and constituting an angle ϕ with the positive direction of x -axis. Then h is an involution preserving the unit circle and the restriction $h|_{S^1} : S^1 \rightarrow S^1$ is a map of degree -1 belonging to $SO^-(2)$. Moreover, each its lifting $h_a : \mathbb{R} \rightarrow \mathbb{R}$ is given by $\tilde{h}_a(s) = a + \phi - s$. But then $\tilde{h}_a^2 = \text{id}_{\mathbb{R}^2}$ and does not depend on a . Moreover, the set of fixed points of $\tilde{h}_a^2$ is $\mathbb{R}$ which coincides with $p^{-1}(S^1)$, where S^1 is the set of fixed points of $h^2 = \text{id}_{S^1}$.

Another interpretation of the above results can be given in terms of *shift functions*.

Corollary 5.3. Let $\mathbf{F} : S^1 \times \mathbb{R} \rightarrow S^1$ be a flow on the circle S^1 having no fixed points, so S^1 is a unique periodic orbit of $\mathbf{F}$ of some period θ .

(1) Let $h : S^1 \rightarrow S^1$ be a continuous map. Then $h \in C_1(S^1, S^1)$ if and only if there exists a continuous function $\alpha : S^1 \rightarrow \mathbb{R}$ such that $h = \mathbf{F}_\alpha$. Such a function is not unique and is determined up to a constant summand $n\theta$ for $n \in \mathbb{Z}$. If $\mathbf{F}$ and h are C^r , ($0 \leq r \leq \infty$), then so is α .

(2) For every $h \in C_{-1}(S^1, S^1)$ there exists a unique continuous function $\alpha : S^1 \rightarrow \mathbb{R}$ such that

- (a) $h^2(z) = \mathbf{F}(z, \alpha(z))$ for all $z \in S^1$;
- (b) $\alpha(z) = 0$ for some $z \in S^1$ iff $h(z) = z$, (due to Lemma 5.1(i) there exists at least two such points);
- (c) if $\mathbf{F}$ and h are C^r , ($0 \leq r \leq \infty$), then so is α .

(3) Let $\alpha_k : S^1 \rightarrow \mathbb{R}$, $k = 0, 1$, be two continuous functions, and for each $t \in [0; 1]$ let $\alpha_t : S^1 \rightarrow \mathbb{R}$ and $h_t : S^1 \rightarrow S^1$ be defined by

$$\alpha_t = (1 - t)\alpha_0 + t\alpha_1, \quad h_t(z) = ze^{2\pi i\alpha_k(z)}.$$

If h_0 and h_1 are homeomorphisms (diffeomorphisms of class $\mathcal{C}^r$, $1 \leq r \leq \infty$), then so is h_t for each $t \in [0; 1]$.

Proof. Let $\tilde{\mathbf{F}} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a unique lifting of $\mathbf{F}$ with respect to the covering p . Then $\mathbb{R}$ is a unique non-periodic orbit of $\tilde{\mathbf{F}}$.

(1) If $\alpha : S^1 \rightarrow \mathbb{R}$ is any continuous function, then the map $\mathbf{F}_\alpha$ is homotopic to the identity by the homotopy $\{\mathbf{F}_{t\alpha}\}_{t \in [0; 1]}$, whence $\mathbf{F}_\alpha$ has degree 1 (the same as id_{S^1}).

Conversely, let $h \in C_1(S^1, S^1)$ and let $\tilde{h}$ be any lifting of h . Then by Corollary 4.6 there exists a unique continuous function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\tilde{h} = \tilde{\mathbf{F}}_\beta$. Moreover, since h is a map of degree 1, we get from Lemma 5.1(b), that $\tilde{h}$ commutes with ξ , i.e. condition (g3) of Lemma 4.7 holds. Since $\tilde{h} = \mathbf{F}_\beta$ is a lifting of h , Lemma 4.7(ii) implies that condition (g1) of Lemma 4.7 also holds, i.e. $\beta \circ \xi = \beta$. Hence β induces a unique function $\alpha : S^1 \rightarrow \mathbb{R}$ such that $\beta = \alpha \circ p$ and $\tilde{h} = \tilde{\mathbf{F}}_{\alpha \circ p}$. Therefore By Lemma 4.7(2), $\tilde{h}$ is a lifting of $\mathbf{F}_\alpha$. But it is also a lifting of h , whence $h = \mathbf{F}_\alpha$.

(2) Since $h \in C_{-1}(S^1, S^1)$, it follows that $h^2 \in C_1(S^1, S^1)$, whence (a) and (c) directly follow from (1).

(b) Denote by A the set of fixed points of h^2 . We should prove that $A = \alpha^{-1}(0)$. Evidently, if $\alpha(y) = 0$, then

$$h^2(y) = \mathbf{F}(y, \alpha(y)) = h^2(y) = \mathbf{F}(y, 0) = y.$$

Conversely, let $y \in A$, so $h(y) = y$, and let $x \in \mathbb{R}$ be any point with $p(x) = y$. Then there exists a unique lifting $\tilde{h}$ of h with $\tilde{h}(x) = x$ and by Lemma 5.1(iv), $p^{-1}(A)$ is the set of fixed points of $\tilde{h}^2$. Moreover, since $\tilde{h}^2$ has degree 1, we get from (1) that $\tilde{h}^2 = \tilde{\mathbf{F}}_{\alpha \circ p}$. Since x is a fixed point of $\tilde{h}^2$ as well, we must have that $0 = \alpha \circ p(x) = \alpha(y)$.

(3) Consider two flows of $\mathbb{R}$ and S^1 respectively:

$$\begin{aligned} \tilde{\mathbf{F}} : \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R}, & \tilde{\mathbf{F}}(x, t) &= x + t, \\ \mathbf{F} : S^1 \times \mathbb{R} &\rightarrow S^1, & \mathbf{F}(z, t) &= ze^{2\pi it}. \end{aligned}$$

Evidently, $\tilde{\mathbf{F}}$ is a lifting of $\mathbf{F}$, and $h_t = \mathbf{F}_{\alpha_t}$. Then by Lemma 4.7(2) the map

$$\tilde{h}_t := \tilde{\mathbf{F}}_{\alpha_t \circ p} : \mathbb{R} \rightarrow \mathbb{R}, \quad \tilde{h}_t(x) = x + (1 - t)\alpha_0(x) + t\alpha_1(x)$$

is a lifting of h_t . Evidently, $\tilde{h}_t = (1 - t)\tilde{h}_0 + t\tilde{h}_1$. Since h_0 and h_1 are homeomorphisms (diffeomorphisms of class $\mathcal{C}^r$), it follows that so are $\tilde{h}_0$ and

$\tilde{h}_1$ are homeomorphisms. Therefore so are their *convex linear combination* $\tilde{h}_t$ and the induced map h_t . $\square$

6. FLOWS WITHOUT FIXED POINTS

Let X be a Hausdorff topological space and $\mathbf{F}: Y \times \mathbb{R} \rightarrow Y$ a continuous flow on an open subset $Y \subset X \times S^1$ satisfying the following conditions:

- ($\Phi 1$) *the orbits of $\mathbf{F}$ are exactly the connected components of the intersections $(x \times S^1) \cap Y$ for all $x \in X$.*
- ($\Phi 2$) *$\mathbf{F}$ admits flow box charts at each point $(y, s) \in Y$;*
- ($\Phi 3$) *the set $B = \{x \in X \mid x \times S^1 \subset Y\}$ is dense in X ;*
- ($\Phi 4$) *for each $x \in X$, the intersection $(x \times S^1) \cap Y$ has only finitely many connected components (being by ($\Phi 1$) orbits of $\mathbf{F}$).*

Thus every orbit of $\mathbf{F}$ is either $x \times S^1$ or some arc in $x \times S^1$, and, in particular, $\mathbf{F}$ has no fixed points. Also condition ($\Phi 3$) implies that the set of periodic orbits of $\mathbf{F}$ is dense in $X \times S^1$. Then the complement to B :

$$A := X \setminus B = \{x \in X \mid x \times S^1 \not\subset Y\}$$

consists of $x \in X$ for which $x \times S^1$ contains a non-closed orbit of $\mathbf{F}$.

It will also be convenient to use the following notations for $x \in X$:

$$L_x := (x \times S^1) \cap Y, \quad \tilde{L}_x := p^{-1}(L_x) = (x \times \mathbb{R}) \cap \tilde{Y}.$$

Example 6.1. Let X be a smooth manifold, and G be a vector field on $X \times S^1$ defined by $G(x, s) = \frac{\partial}{\partial s}$, so its orbits are the circles $x \times S^1$. Let $Y \subset X \times S^1$. Then it is well known that there exists a non-negative $\mathcal{C}^\infty$ function $\alpha: X \times S^1 \rightarrow [0; +\infty)$ such that $Y = (X \times S^1) \setminus \alpha^{-1}(0)$. Define another vector field F on $X \times S^1$ by $F = \alpha G$. Let also

$$\mathbf{F}: (X \times S^1) \times \mathbb{R} \rightarrow X \times S^1$$

be the flow on $X \times S^1$ generated by F . Then Y is the set of non-fixed points of $\mathbf{F}$ and the induced flow on Y satisfies conditions ($\Phi 1$) and ($\Phi 2$). Another two conditions ($\Phi 3$) and ($\Phi 4$) depend only on a choice of Y . For instance they will hold if the complement $(X \times S^1) \setminus Y$ is finite.

Theorem 6.2. *Suppose $\mathbf{F}$ satisfies ($\Phi 1$)-($\Phi 4$). Let also $h: Y \rightarrow Y$ be a homeomorphism having the following properties.*

- (1) *$h(L_x) \subset x \times S^1$ for each $x \in X$, so h preserves the first coordinate, and in particular, leaves invariant each periodic orbit of $\mathbf{F}$, though it may interchange non-periodic orbits contained in $x \times S^1$.*
- (2) *For each $x \in B$ the restriction $h: x \times S^1 \rightarrow x \times S^1$ has degree -1 as a self-map of a circle.*

Then there exists a unique continuous function $\alpha: Y \rightarrow \mathbb{R}$ such that

- (a) $h^2(x, s) = \mathbf{F}(x, s, \alpha(x, s))$ for all $(x, s) \in Y$, in particular h^2 preserves **every** orbit of $\mathbf{F}$;
- (b) $\alpha(x, s) = 0$ iff $h(x, s) = (x, s)$;
- (c) if in addition X is a manifold of class C^r , $(0 \leq r \leq \infty)$, $\mathbf{F}$ is C^r and admits C^r flow box charts, and h is C^r , then α is C^r as well.

Proof. Let $p: X \times \mathbb{R} \rightarrow X \times S^1$ be the infinite cyclic covering of $X \times S^1$ defined by $p(x, s) = (x, e^{2\pi i s})$ and $\xi(x, s) = (x, s + 1)$ be a diffeomorphism of $X \times \mathbb{R}$ generating the group $\mathbb{Z}$ of covering slices. In particular, $p \circ \xi = p$. Denote $\tilde{Y} = p^{-1}(Y)$. Then the restriction $p: \tilde{Y} \rightarrow Y$ is a covering of Y .

Let $\tilde{\mathbf{F}}: \tilde{Y} \times \mathbb{R} \rightarrow \tilde{Y}$ be the lifting of the flow $\mathbf{F}$, so

$$p \circ \tilde{\mathbf{F}}(y, t) = \mathbf{F}(p(y), t), \quad (y, t) \in \tilde{Y} \times \mathbb{R}.$$

Then the orbits of $\tilde{\mathbf{F}}$ are exactly the connected components of $\tilde{L}_x$ for all $x \in X$. In particular, all orbits of $\tilde{\mathbf{F}}$ are non-closed. Evidently, for $x \in X$ the following conditions are equivalent:

- $x \times \mathbb{R}$ is an orbit of $\tilde{\mathbf{F}}$;
- $x \times \mathbb{R} \subset \tilde{Y}$;
- $x \times S^1$ is an orbit of $\mathbf{F}$;
- $x \times S^1 \subset Y$, i.e. $x \in B$.

Lemma 6.2.1. *Let $\tilde{h}: \tilde{Y} \rightarrow \tilde{Y}$ be any lifting of h , that is $p \circ \tilde{h} = h \circ p$. Then the following conditions hold.*

- (i) $\tilde{h}(\tilde{L}_x) \subset \tilde{L}_x$ for all $x \in X$;
- (ii) if $\tilde{h}_1$ is another lifting of h , then $\tilde{h}_1^2 = \tilde{h}^2$;
- (iii) for each $x \in X$ the restriction $\tilde{h}: \tilde{L}_x \rightarrow \tilde{L}_x$ is strictly decreasing in the sense that if $h(x, s) = (x, t)$ for some $s, t \in \mathbb{R}$, then $s > t$;
- (iv) for each $x \in X$ the restriction $\tilde{h}^2$ leaves invariant each orbit of $\tilde{\mathbf{F}}$.

Proof. (i) Notice that $p \circ \tilde{h}(\tilde{L}_x) = h \circ p(\tilde{L}_x) = h(L_x) \subset L_x$, whence

$$\tilde{h}(\tilde{L}_x) \subset p^{-1}(h(L_x)) \subset p^{-1}(L_x) = \tilde{L}_x.$$

(ii) Let $x \in B$, so $x \times S^1 \subset Y$ is an orbit of $\mathbf{F}$. Then the restriction $p|_{x \times \mathbb{R}}: x \times \mathbb{R} \rightarrow x \times S^1$ is a universal covering map, $h|_{x \times S^1}: x \times S^1 \rightarrow x \times S^1$ is a map of degree -1 , and $\tilde{h}|_{x \times \mathbb{R}}, \tilde{h}_1|_{x \times \mathbb{R}}: x \times \mathbb{R} \rightarrow x \times \mathbb{R}$ are two liftings of $h|_{x \times S^1}$. Hence by Lemma 5.1(iv) $\tilde{h}^2|_{x \times \mathbb{R}} = \tilde{h}_1^2|_{x \times \mathbb{R}}$. Thus $\tilde{h}^2 = \tilde{h}_1^2$ on $B \times \mathbb{R}$. But by property $(\Phi 3)$, B is dense in X , whence $B \times \mathbb{R}$ is dense in $X \times \mathbb{R}$. Since X is Hausdorff, it follows that $\tilde{h}^2 = \tilde{h}_1^2$ on all of $X \times \mathbb{R}$.

(iii) If $x \in B$, then $h|_{x \times S^1}: x \times S^1 \rightarrow x \times S^1$ is a map of degree -1 , whence $\tilde{h}|_{x \times \mathbb{R}}: x \times \mathbb{R} \rightarrow x \times \mathbb{R}$ reverses orientation of $x \times \mathbb{R}$, i.e. is strictly decreasing.

Suppose $x \in X \setminus B$, i.e. $x \times \mathbb{R} \not\subset \tilde{Y}$. We need to show that $\tilde{h}|_{\tilde{L}_x} : \tilde{L}_x \rightarrow \tilde{L}_x$ is strictly decreasing, that is if $(x, s_0), (x, s_1) \in \tilde{L}_x$ are two distinct points with $s_0 < s_1$, $(x, t_0) = \tilde{h}(x, s_0)$, and $(x, t_1) = \tilde{h}(x, s_1)$, then $t_0 > t_1$.

Since $\tilde{h}$ is a homeomorphism, it follows that $t_0 \neq t_1$. Suppose that $t_0 < t_1$. Then there exist $a > 0$, and two open neighborhoods $V \subset U$ of x in X such that

$$\tilde{h}(V \times s_i) \subset U \times (t_i - a; t_i + a), \quad i = 0, 1, \quad (6.1)$$

$$t_0 + a < t_1 - a. \quad (6.2)$$

Due to property $(\Phi 3)$, the set B is dense in X , so there exists a point $y \in B \cap V \neq \emptyset$.

Then on the one hand $y \times \mathbb{R} \subset \tilde{Y}$ is an orbit of $\tilde{\mathbf{F}}$, and $\tilde{h} : y \times \mathbb{R} \rightarrow y \times \mathbb{R}$ reverses orientation, so if $(y, t'_i) = \tilde{h}(y, s_i)$, $i = 0, 1$, then $t'_0 > t'_1$.

On the other hand, due to (6.1) $t'_i \in (t_i - a; t_i + a)$, whence by (6.2):

$$t'_0 < t_0 + a < t_1 - a < t'_1$$

which gives a contradiction. Hence $t_0 > t_1$.

(iv) If $x \in B$, then $\tilde{L}_x = x \times \mathbb{R}$ is an orbit of $\tilde{\mathbf{F}}$ and by (i) it is invariant with respect to $\tilde{h}$. Hence it is also invariant with respect to $\tilde{h}^2$.

Suppose $x \in X \setminus B$. Then by property $(\Phi 4)$, if $L_x = (x \times S^1) \cap Y$ consists of finitely many connected components $I_0, \dots, I_{n-1}$ for some n enumerated in the cyclical order along the circle $x \times S^1$. This implies that $\tilde{L}_x$ is a disjoint union of countably many open intervals I_i , ($i \in \mathbb{Z}$), being orbits of $\mathbf{F}$ which can be enumerated so that $\xi(\tilde{I}_k) = \tilde{I}_{k+n}$ and $p(\tilde{I}_k) = I_{k \bmod n}$.

Since $\tilde{h}$ is a strictly decreasing homeomorphism of $\tilde{L}_x$, it follows that there exists $a \in \mathbb{Z}$ such that $\tilde{h}(\tilde{I}_k) = \tilde{I}_{a-k}$. Hence

$$\tilde{h}^2(\tilde{I}_k) = \tilde{h}(\tilde{I}_{a-k}) = \tilde{I}_{a-(a-k)} = \tilde{I}_k. \quad \square$$

Now we can deduce our Theorem from Lemma 6.2.1.

(a) By Corollary 4.8 applied to h^2 there exists a unique continuous function $\alpha : Y \rightarrow \mathbb{R}$ such that

$$\tilde{h}^2(x, s) = \tilde{\mathbf{F}}(x, s, \alpha \circ p(x, s)), \quad h^2(x, s) = \mathbf{F}(x, s, \alpha(x, s)).$$

(b) Let $(x, s) \in X \times S^1$. If $x \in B$, then $x \times S^1 \subset Y$ is an orbit of $\mathbf{F}$ and by Corollary 5.3(b) $\alpha(x, s) = 0$ iff $h^2(x, s) = (x, s)$.

Suppose $x \in X \setminus B$. Then (x, s) is a non-periodic point of $\mathbf{F}$, whence $h^2(x, s) = \mathbf{F}(x, s, \alpha(x, s)) = (x, s)$ is possible if and only if $\alpha(x, s) = 0$.

(c) Smoothness properties of α follows similarly to the statement (c) of Corollary 4.8. $\square$

7. POLAR COORDINATES

Let $\mathbb{H} = \mathbb{R} \times [0; +\infty)$, $\text{Int}(\mathbb{H}) = \mathbb{R} \times (0; +\infty)$, and

$$p : \mathbb{H} \rightarrow \mathbb{C} \equiv \mathbb{R}^2, \quad p(\rho, \phi) = \rho e^{2\pi i \phi} = (\rho \cos \phi, \rho \sin \phi),$$

be the *infinite branched covering map defining polar coordinates*.

Lemma 7.1. ([9, Lemma 11.1]) *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, and $h : U \rightarrow \mathbb{C}$ a $\mathcal{C}^r$, $1 \leq r \leq \infty$, smooth embedding with $h(0) = 0$. Then there exists a $\mathcal{C}^{r-1}$ embedding $\tilde{h} : p^{-1}(U) \rightarrow \mathbb{H}$ such that $p \circ \tilde{h} = h \circ p$.*

Suppose, in addition, that the Jacobi matrix $J(h, 0)$ of h at 0 is orthogonal. Then there exists $a \in \mathbb{R}$ such that for each $s \in \mathbb{R}$

$$\tilde{h}(0, s) = \begin{cases} (0, a + s) & \text{if } J(h, 0) \in \text{SO}(2), \\ (0, a - s) & \text{if } J(h, 0) \in \text{SO}^-(2). \end{cases}$$

Hence in the second case (when h reverses orientation) $\tilde{h}^2(0, s) = (0, s)$, that is $\tilde{h}$ is always fixed on $0 \times \mathbb{R}$.

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a homogeneous polynomial without multiple factors and having degree $k \geq 2$ and $F = -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$ be the Hamiltonian vector field of f . Since the restriction $p : \text{Int}(\mathbb{H}) \rightarrow \mathbb{R}^2 \setminus 0$ is a infinite cyclic covering map, F induces a vector field $\tilde{F}$ on $\text{Int}(\mathbb{H})$. One can even obtain precise formulas for $\tilde{F}$ (see [6, §4.2, Corolary 4.4]):

$$\tilde{F}(r, \phi) = \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi} \right) \begin{pmatrix} \cos \phi & \sin \phi \\ -\frac{1}{\rho} \sin \phi & \frac{1}{\rho} \cos \phi \end{pmatrix} \begin{pmatrix} -f'_y(z) \\ f'_x(z) \end{pmatrix},$$

where $z = p(\rho, \phi) = \rho e^{2\pi i}$. The latter formula can be reduced to a more simplified form. Define the following function $q : \mathbb{C} \setminus 0 \rightarrow \mathbb{C}$ by

$$q(z) = \frac{-f'_y(z) + i f'_x(z)}{z} = \frac{(-f'_y(z) + i f'_x(z)) \bar{z}}{|z|^2}.$$

Then

$$\tilde{F}(r, \phi) = \text{Re}(q(z)) \frac{\partial}{\partial \rho} + \text{Im}(q(z)) \frac{1}{\rho} \frac{\partial}{\partial \phi}.$$

Since f is a homogeneous of degree $k \geq 2$, it follows that $\tilde{F}(r, \phi)$ smoothly extends to $\mathbb{H}$.

Let $\mathbf{F}$ and $\tilde{\mathbf{F}}$ be the local flows generated by F and $\tilde{F}$ respectively. Then $\mathbf{F}_t \circ p(\rho, \phi) = p \circ \tilde{\mathbf{F}}_t(\rho, \phi)$ whenever all parts of that identity are defined.

Example 7.2. 1) Suppose 0 is a non-degenerate local extreme of f . Then one can assume that $f(x, y) = \frac{1}{2}(x^2 + y^2)$. In this case

$$F(x, y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad \mathbf{F}(z, t) = z e^{2\pi i t},$$

$$\tilde{F}(\rho, \phi) = \frac{\partial}{\partial \phi},$$

$$\tilde{\mathbf{F}}(\rho, \phi, t) = (\rho, \phi + t).$$

2) Let 0 be a non-degenerate saddle, so one can assume that $f(x, y) = xy$. Then

$$F(x, y) = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad \mathbf{F}(x, y, t) = (xe^{-t}, ye^t),$$

$$\tilde{F}(\rho, \phi) = \rho \cos 2\phi \frac{\partial}{\partial \rho} + \sin 2\phi \frac{\partial}{\partial \phi}.$$

Notice that writing down precise formulas for $\tilde{\mathbf{F}}$ is a rather complicated task.

3) If 0 is a degenerate critical point of f , so $\deg f \geq 3$, then the situation is more complicated. Notice that in this case $\tilde{F}$ is zero on $\partial\mathbb{H} = \mathbb{R} \times 0$, whence $\tilde{\mathbf{F}}$ is fixed on that line. Again the formulas for $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are highly complicated.

Lemma 7.3. (see [6, Theorem 1.6], [9, Proof of Theorem 5.6]) *Let $U \subset \mathbb{R}^2$ be an open neighborhood of the origin $0 \in \mathbb{R}^2$, $h : U \rightarrow \mathbb{R}^2$ an embedding which preserves orbits of $\mathbf{F}$, and $\alpha : U \setminus 0 \rightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $h(x) = \mathbf{F}(x, \alpha(x))$ for all $x \in U \setminus 0$. Let also $\tilde{h} : p^{-1}(U) \rightarrow \mathbb{H}$ be any lifting of h as in Lemma 7.1. Then α can be defined at x so that it becomes $\mathcal{C}^\infty$ in U in the following cases:*

- (a) *x is a non-degenerate local extreme of f or a (possibly degenerate) saddle point;*
- (b) *x is a degenerate local extreme of f and $\tilde{h}$ is fixed on $\mathbb{R} \times 0$.*
- (c) *x is a degenerate local extreme of f and there exists an open neighborhood $V \subset U$ and another embedding $q : U \rightarrow \mathbb{R}^2$ such that $q(V) \subset U$, q preserves orbits of $\mathbf{F}$ and **reverses** their orientation, and $h = q^2$.*

Proof. Cases (a) and (b) are proved in [6, Theorem 1.6], see also proof of Theorem 5.6 in [9].

(c) $\Rightarrow$ (b) Let $\tilde{q} : p^{-1}(U) \rightarrow \mathbb{H}$ be any lifting of q as in Lemma 7.1. Then $\tilde{q}^2$ is a lifting of $q^2 = h$. Moreover, since $\tilde{q}$ reverse orientation of orbits, we get from Lemma 7.1 that $\tilde{q}^2$ is fixed on $\mathbb{R} \times 0$. Hence condition (b) holds. $\square$

8. CHIPPED CYLINDERS OF A MAP $f \in \mathcal{F}(M, P)$

Let $f \in \mathcal{F}(M, P)$. In what follows we will use the following notations.

- (i) $K_1, \dots, K_k$ denote all the critical leaves of f , and

$$\mathcal{K} = \bigcup_{i=1}^k K_i.$$

- (ii) Let R_{K_i} , $i = 1, \dots, k$, be an f regular neighborhood of K_i chosen so that $R_{K_i} \cap R_{K_j} = \emptyset$ for $i \neq j$.

- (iii) Let also $L_1, \dots, L_l$ be all the connected components of $M \setminus \mathcal{K}$;
- (iv) For each $i = 1, \dots, l$ let

$$\mathcal{N}_i = \overline{L_i} \setminus \Sigma_f.$$

Then there exist a finite subset $Q_i \subset \{-1, 1\} \times S^1$, and an immersion $\phi_i : ([-1, 1] \times S^1) \setminus Q_i \rightarrow \mathcal{N}_i$ and a $\mathcal{C}^\infty$ embedding $\eta : [0, 1] \rightarrow P$ such that the following diagram is commutative:

$$\begin{array}{ccc} ([0; 1] \times S^1) \setminus Q_i & \xrightarrow{\phi} & \mathcal{N}_i \\ p_1 \downarrow & & \downarrow f \\ [0; 1] & \xrightarrow{\eta} & P \end{array}$$

where p_1 is the projection to the first coordinate. Notice that ϕ can be non-injective only at points of $\{-1, 1\} \times S^1$ and this can happens only when $P = S^1$, see Example 8.1 and Figure 8.2d) below.

We will call $\mathcal{N}_i$ a *chipped cylinder* of f , see Figure 8.1.

FIGURE 8.1.

It will also be convenient to denote

$$\begin{aligned} \mathcal{N}_i^- &= \phi_i([-1; 0] \times S^1) \setminus Q_i, & \mathcal{N}_i^+ &= \phi_i([0; 1] \times S^1) \setminus Q_i, \\ \text{Int } \mathcal{N}_i &= \phi_i((-1; 1) \times S^1). \end{aligned}$$

We will call $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$ *chipped half-cylinders* of $\mathcal{N}_i$ and f , and $\text{Int } \mathcal{N}_i$ the *interior* of $\mathcal{N}_i$.

- (v) Let also

$$Z_i = \mathcal{N}_i \bigcup \left(\bigcup_{\overline{\mathcal{N}_i} \cap K_j \neq \emptyset} R_{K_j} \right)$$

be the union of the chipped cylinder $\mathcal{N}_i$ with f -regular neighborhoods of critical leaves of f which intersect the closure $\overline{\mathcal{N}_i}$. We will call Z_i an *f -regular neighborhood* of $\mathcal{N}_i$.

Example 8.1. a) Let $f : [0, 1] \times S^1 \rightarrow \mathbb{R}$ be given by $f(s, x) = s$, so it has no critical points, see Figure 8.2a). Then it has a unique chipped cylinder $\mathcal{N} = [0, 1] \times S^1$, which coincides with its f -regular neighborhood and $\mathcal{K} = \emptyset$.

b) Let $f : D^2 \rightarrow \mathbb{R}$ be a function of class $\mathcal{F}(D^2, \mathbb{R})$ having only one critical point z , see Figure 8.2b). Then $\mathcal{K} = \{z\}$, f has a unique chipped cylinder $\mathcal{N} = D^2 \setminus \{z\}$, and its f -regular neighborhood is all D^2 .

c) Let $f : S^2 \rightarrow \mathbb{R}$ be a function of class $\mathcal{F}(S^2, \mathbb{R})$ having only two critical points z_1 and z_2 being therefore extremes of f , see Figure 8.2c). Then $\mathcal{K} = \{z_1, z_2\}$, f has a unique chipped cylinder $\mathcal{N} = S^2 \setminus \{z_1, z_2\}$ and its f -regular neighborhood is all S^2 .

d) Let M be either a 2-torus or Klein bottle with a hole and $f : M \rightarrow S^1$ be a map of class $\mathcal{F}(M, P)$ schematically shown in Figure 8.2d). It has only one critical point z and that point is a saddle, a unique critical leaf $K = \mathcal{K}$, and two chipped cylinders $\mathcal{N}_1$ and $\mathcal{N}_2$. It follows from the Figure 8.2d) that $\overline{\mathcal{N}_1}$ intersects only one K from “both sides”, in the sense that both intersections $\overline{\mathcal{N}_1}^- \cap K$ and $\overline{\mathcal{N}_1}^+ \cap K$ are non-empty.

e) Let $f : [0, 1] \times S^1 \rightarrow \mathbb{R}$ be a Morse function having one minimum z and one saddle point y as in Figure 8.3. Then f has two critical leaves: the point z and a critical leaf K containing y , and three chipped cylinders $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$. Let R_K be an f -regular neighborhood of K . Then the corresponding f -regular neighborhoods of chipped cylinders are the following ones:

$$Z_1 = \mathcal{N}_1 \cup \{z\} \cup R_K, \quad Z_2 = \mathcal{N}_2 \cup R_K, \quad Z_3 = \mathcal{N}_3 \cup R_K.$$

FIGURE 8.2.

FIGURE 8.3.

The following lemma describes simple properties of chipped cylinders. The proof is left for the reader.

Lemma 8.2. *Let $\mathcal{N}$ be a chipped cylinder of f , $\mathcal{N}^-$ and $\mathcal{N}^+$ be its chipped half-cylinders, and $p : M \rightarrow \Gamma_f$ be the projection onto the graph of f . Then the following statements hold.*

- (1) $\mathcal{N}^-$ and $\mathcal{N}^+$ are orientable manifolds. Moreover, if $P = \mathbb{R}$, then $\mathcal{N}$ is orientable as well. However, if $P = S^1$, then it is possible to construct an example of f having non-orientable chipped cylinder, see Example 8.1d).
- (2) Each of the closures $\overline{\mathcal{N}^-}$ and $\overline{\mathcal{N}^+}$ intersects at most one critical leaves of f , and those intersections consist of open arcs being leaves of the singular foliation Ξ_f .
- (3) If $\mathcal{N} \cap K_i = \emptyset$ but $\overline{\mathcal{N}} \cap K_i \neq \emptyset$, then K_i is a critical point of f being a local extreme.
- (4) $\mathcal{N}' \cap \mathcal{N}'' \subset \mathcal{K}$ for any two distinct chipped cylinders $\mathcal{N}'$ and $\mathcal{N}''$ of f .
- (5) Every regular leaf of f is contained in some chipped cylinder of f .

The following theorem is the principal technical tool. Let M be a (possibly non-orientable) compact surface, $f \in \mathcal{F}(M, P)$, $\mathcal{N}$ a chipped cylinder of f , and $Z = \mathcal{N} \cup \left(\bigcup_{\overline{\mathcal{N}} \cap K_j \neq \emptyset} R_{K_j} \right)$ be its f -regular neighborhood of $\mathcal{N}$.

Theorem 8.3. 1) *Let $V \subset \mathcal{N}$ be a regular leaf of f , and $h \in \mathcal{S}(f|_Z)$ a diffeomorphism of Z such that $h(V) = V$ and h **reverses** orientation of V . Then every leaf of the singular foliation Ξ_f in Z is $(h^2, +)$ -invariant, i.e. h^2 fixes all critical points of f in Z and preserves all other leaves of Ξ_f in Z with their orientations.*

2) *Suppose in addition that Z is orientable and let $\mathbf{F}$ be any Hamiltonian like flow of $f|_Z$ on Z . Then there exists a unique C^∞ function $\alpha : Z \rightarrow \mathbb{R}$ such that $h^2 = \mathbf{F}_\alpha$ on Z and $\alpha = 0$ at each fixed point of h^2 in $\text{Int } \mathcal{N}$ and for each local extreme z of f in Z .*

Proof. 1) Let us mention that since h reverses orientation of V , it reverses orientations of all regular leaves in $\mathcal{N}$. Therefore those leaves are $(h^2, +)$ -invariant, and we should prove the same for all other leaves of Ξ_f in Z .

First we introduce the following notation. If $\mathcal{K} \cap \overline{\mathcal{N}^-} \neq \emptyset$, then let K^- be a unique critical leaf of f intersecting $\overline{\mathcal{N}^-}$, and let R_{K^-} be its f -regular neighborhood. Otherwise, when $\mathcal{K} \cap \overline{\mathcal{N}^-} = \emptyset$, put $K^- = R_{K^-} = \emptyset$. Define further K^+ and R_{K^+} in a similar way with respect to $\mathcal{N}^+$. Then

$$Z = R_{K^-} \cup \mathcal{N}^- \cup \mathcal{N}^+ \cup R_{K^+}.$$

As those four sets are invariant with respect to h , it suffices to prove that h^2 preserves leaves of Ξ_f with their orientation for each of those sets.

1a) Let us show that h^2 preserves all leaves of Ξ_f in $\mathcal{N}^-$.

Since $\mathcal{N}^-$ is an orientable manifold, one can construct a Hamiltonian like flow of f on $\mathcal{N}^-$. Evidently, $\mathbf{F}$ satisfies conditions $(\Phi 1)$ – $(\Phi 4)$. Moreover, since h reverses orientation of all periodic orbits of $\mathbf{F}$ in $\mathcal{N}^+$, we get from Theorem 6.2 that there exists a unique $\mathcal{C}^\infty$ function $\alpha : \mathcal{N} \rightarrow \mathbb{R}$ such that $h^2|_{\mathcal{N}^-} = \mathbf{F}_\alpha$ and α vanishes at fixed points of h^2 on regular leaves of f in $\mathcal{N}^-$. In particular, each non-periodic orbit of $\mathbf{F}$ in $\mathcal{N}^+$ is $(h^2, +)$ -invariant as well.

1b) Now let us prove that each leaf of Ξ_f in K^- is also $(h^2, +)$ -invariant. This will imply $(h^2, +)$ -invariantness of all leaves of Ξ_f in R_{K^-} (see the proof of the implication (iii) $\Rightarrow$ (i) in [9, Lemma 7.4]).

If $K^- = \emptyset$ there is nothing to prove.

If $K^- \cap \mathcal{N}^- = \emptyset$, then by Lemma 8.2(3) K^- is a local extreme of f . Hence K is an element of Ξ_f and its is evidently invariant with respect h , and therefore with respect to h^2 .

Thus assume that $K^- \cap \mathcal{N}^- \neq \emptyset$. Then this intersection contains a non-periodic orbit γ of $\mathbf{F}$. Since γ is $(h^2, +)$ -invariant, it follows from [7, Claim 7.1.1] or [9, Lemma 7.4], that *all elements of the foliation Ξ_f are also $(h^2, +)$ -invariant*.

Let us recall a simple proof of that fact. Indeed, let v be a vertex of γ being therefore a critical point of f . Then $h(v) = v$, whence h^2 preserves the set of all edges incident to v . Moreover, as h^2 preserves orientation at v , it must also *preserve cyclic order of edges incoming to v* . But since γ (being one of those edges) is $(h^2, +)$ -invariant, it follows that so are all other edges incident to v . Applying the same arguments to those edges and so on, we will see that h^2 preserves all edges of K with their orientation.

The proofs for $\mathcal{N}^+$ and R_{K^+} are similar.

2) Assume now that Z is an orientable surface and let $\mathbf{F}$ be any Hamiltonian like flow of f . We know from 1) that h^2 preserves all orbits of $\mathbf{F}$ with their orientations.

2a) We claim that *there exists a unique $\mathcal{C}^\infty$ function $\alpha : \mathcal{N} \rightarrow \mathbb{R}$ such that $h^2|_{\mathcal{N}} = \mathbf{F}_\alpha$ and α vanishes at fixed points of h^2 on periodic orbits*.

If both K^- and K^+ are non-empty and distinct, then the restriction of $\mathbf{F}$ to $\mathcal{N}$ satisfies conditions $(\Phi 1)$ – $(\Phi 4)$. Moreover, as h reverses orientation of all periodic orbits of $\mathbf{F}$, the statement follows from Theorem 6.2 as in 1a).

However, if $K^- = K^+$, as in Example 8.1d), the situation is slightly more complicated: $\mathcal{N}$ might be not of the form $([-1, 1] \times S^1) \setminus Q$ for some finite set $Q \subset \{-1, 1\} \times S^1$ and Theorem 6.2 is not directly applicable. Nevertheless, one can apply that theorem to each of the sets $\mathcal{N}^-$, $\mathcal{N}^+$, and

$\text{Int } \mathcal{N}$ and construct three functions

$$\alpha^- : \mathcal{N}^- \rightarrow \mathbb{R}, \quad \alpha^+ : \mathcal{N}^+ \rightarrow \mathbb{R}, \quad \alpha^0 : \text{Int } \mathcal{N} \rightarrow \mathbb{R}$$

satisfying $h^2|_{\mathcal{N}^-} = \mathbf{F}_{\alpha^-}$, $h^2|_{\mathcal{N}^+} = \mathbf{F}_{\alpha^+}$, $h^2|_{\text{Int } \mathcal{N}} = \mathbf{F}_{\alpha^0}$, and vanishing at fixed points of h^2 on periodic orbits. From uniqueness of such functions, we get that $\alpha^- = \alpha^0$ on $\mathcal{N}^- \cap \text{Int } \mathcal{N}$ and $\alpha^+ = \alpha^0$ on $\mathcal{N}^+ \cap \text{Int } \mathcal{N}$.

A possible problem is that $\mathcal{N}$ intersects K^- from “both sides”, and therefore a priori α^+ and α^- can differ on $\mathcal{N}^- \cap \mathcal{N}^+ \cap K^-$. However, $\mathcal{N}^- \cap \mathcal{N}^+ \cap K^-$ consists of non-periodic orbits of $\mathbf{F}$, and therefore for each such orbit γ the identity $h^2|_{\gamma} = \mathbf{F}_{\alpha^-}|_{\gamma} = \mathbf{F}_{\alpha^+}|_{\gamma}$ implies that $\alpha^- = \alpha^+$ on γ .

Thus $\alpha^- = \alpha^+$ on $\mathcal{N}^- \cap \mathcal{N}^+ \cap K^-$, and therefore those functions define a well defined $\mathcal{C}^\infty$ function $\alpha : \mathcal{N} \rightarrow \mathbb{R}$ satisfying $h^2|_{\text{Int } \mathcal{N}} = \mathbf{F}_\alpha$ and α vanishes at fixed points of h^2 on periodic orbits.

2b) It remains to show that α extends to a shift function for h^2 on $R_{K^-} \cup R_{K^+}$ and thus on all of Z . It suffices to prove that for R_{K^-} .

If $K^- = \emptyset$, then $R_{K^-} = \emptyset$ and there is nothing to prove.

If K^- is a local extreme of f , then by Lemma 7.3 (cases (a) and (c) for non-degenerate and degenerate critical point) α can be defined at K^- so that it becomes $\mathcal{C}^\infty$.

In all other cases K^- contains a non-periodic orbit of $\mathbf{F}$. Then by the implication (ii) $\Rightarrow$ (iv) of [9, Lemma 7.4], α extends to a $\mathcal{C}^\infty$ shift function for h^2 on R_{K^-} . It remains to prove the following lemma:

Lemma 8.3.1. $\alpha(z) = 0$ for every local extreme of f in Z .

Proof. Indeed, it is evident, that arbitrary small neighborhood of z contains a periodic orbit γ of $\mathbf{F}$. Since h reverses orientations of γ , we have from Lemma 5.1(i) that h always has at least one fixed point $x \in \gamma$ (in fact it has even two such points). Hence by Corollary 5.3(b), $\alpha(x) = 0$. Then by continuity of α we should have that $\alpha(z) = 0$ as well. $\square$

Theorem 8.3 is completed. $\square$

9. CREATING ALMOST PERIODIC DIFFEOMORPHISMS

Let M be a compact orientable surface, $f \in \mathcal{F}(M, P)$, Z be an f -adapted subsurface, $h \in \mathcal{S}(f)$ be such that $h(Z) = Z$, and $m \geq 2$. If $h^m|_Z$ is isotopic to the identity of Z by f -preserving isotopy, then the following Lemma 9.1 gives conditions when one can change h on $M \setminus Z$ so that its m -power will be f -preserving isotopic to the identity on all of M . The proof follows the line of [9, Lemma 13.1(3)] in which M is a 2-disk or a cylinder.

Lemma 9.1. (cf. [9, Lemma 13.1(3)]) *Let M be a compact orientable surface, $f \in \mathcal{F}(M, P)$, Z be an f -adapted subsurface, and $h \in \mathcal{S}(f)$ be such that $h(Z) = Z$. Suppose that the following conditions hold.*

- (1) *Each component of Z contains at least one saddle point of f .*
- (2) *There exists $m \geq 2$ and $a \geq 1$ such that connected components of $\overline{M \setminus Z}$ can be enumerated as follows:*

$$\begin{array}{cccc} Y_{1,0} & Y_{1,1} & \cdots & Y_{1,m-1} \\ Y_{2,0} & Y_{2,1} & \cdots & Y_{2,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ Y_{a,0} & Y_{a,1} & \cdots & Y_{a,m-1} \end{array} \quad (9.1)$$

so that $h(Y_{i,j}) = Y_{i,j+1 \bmod m}$ for all i, j , that is h cyclically shifts columns in (9.1).

- (3) *h^m is isotopic in $\mathcal{S}(f)$ to a diffeomorphism τ fixed on some neighborhood of Z (by [9, Lemma 7.1] this condition holds if there exists $\mathcal{C}^\infty$ function $\alpha : Z \rightarrow \mathbb{R}$ such that $h^m|_Z = \mathbf{F}_\alpha$).*

Then there exists $g \in \mathcal{S}(f)$ such that $g = h$ on Z and $g^m \in \mathcal{S}_{\text{id}}(f)$, that is $g^m = \mathbf{F}_\beta$ for some $\mathcal{C}^\infty$ function $\beta : M \rightarrow \mathbb{R}$.

Proof. Let $Y_j = \bigcup_{i=1}^a Y_{i,j}$, $j = 0, \dots, m-1$, be the union of components from the same column of (9.1). Then $h(Y_j) = Y_{j+1 \bmod m}$. Notice that condition (2) implies that $h^m(Y_{i,j}) = Y_{i,j}$ for all i, j .

We will show that the the desired diffeomorphism $g \in \mathcal{S}(f)$ can be defined by the formula:

$$g(x) = \begin{cases} h(x), & x \in Z \cup Y_0 \cup \cdots \cup Y_{m-2}, \\ \tau^{-1} \circ h(x), & x \in Y_{m-1}. \end{cases}$$

Indeed, by definition $g = h$ on Z . Moreover, as τ is fixed on some neighborhood of Z , it also fixed near $Z \cap Y_{m-1}$. Therefore $g = h = \tau^{-1} \circ h$ near $Z \cap Y_{m-1}$, and so g is a well defined $\mathcal{C}^\infty$ map. It remains to prove that $g^m = \mathbf{F}_\beta \in \mathcal{S}_{\text{id}}(f)$ for some $\mathcal{C}^\infty$ function $\beta : M \rightarrow \mathbb{R}$.

Let $\mathbf{F}$ be a Hamiltonian flow for f . Since τ and h^m are isotopic in $\mathcal{S}(f)$, it follows that $\tau^{-1} \circ h^m \in \mathcal{S}_{\text{id}}(f)$. Hence by Lemma 2.3.5, $\tau^{-1} \circ h^m = \mathbf{F}_\alpha$ for some $\mathcal{C}^\infty$ function $\alpha : M \rightarrow \mathbb{R}$.

Since $\mathcal{S}_{\text{id}}(f)$ is a normal subgroup of $\mathcal{S}(f)$, it follows that

$$h^j \circ (\tau^{-1} \circ h^m) \circ h^{-j} = h^j \circ \tau^{-1} \circ h^{m-j} \in \mathcal{S}_{\text{id}}(f), \quad j = 0, \dots, m-1,$$

as well. Therefore, again by Lemma 2.3.5, $h^j \circ \tau^{-1} \circ h^{m-j} = \mathbf{F}_{\alpha_j}$ for some $\mathcal{C}^\infty$ function $\alpha_j : M \rightarrow \mathbb{R}$.

As τ is fixed on some neighborhood of Z , it follows that for each j

$$\mathbf{F}_{\alpha_j} = h^j \circ \tau^{-1} \circ h^{m-j} = \tau^{-1} \circ h^m = \mathbf{F}_\alpha \text{ on } Z.$$

Then the assumption (1) that every connected component Z' of Z contains a saddle point, implies that $\mathbf{F}$ has a non-closed orbit γ in Z' . Therefore $\alpha = \alpha_j$ on γ . Since Z' is connected, it follows from local uniqueness of shift functions for $\tau^{-1} \circ h^m|_Z$ (see Corollary 4.5) that $\alpha = \alpha_j$ near Z' for all $j = 0, \dots, m-1$. Hence $\alpha = \alpha_j$ near all of Z for all $j = 0, \dots, m-1$.

Thus we obtain a well-defined $\mathcal{C}^\infty$ function $\beta : M \rightarrow \mathbb{R}$ given by:

$$\beta(x) = \begin{cases} \alpha(x), & x \in Z, \\ \alpha_j(x), & x \in Y_j, \ j = 0, \dots, m-1. \end{cases}$$

We claim that $g^m = \mathbf{F}_\beta$.

a) Indeed, if $x \in Z$, then $g^m(x) = h^m(x) = \mathbf{F}_\alpha(x) = \mathbf{F}_\beta(x)$.

b) Also notice that $g(Y_{i,j}) = Y_{i,j+1 \bmod m}$ and $g(Y_j) = Y_{j+1 \bmod m}$. Then $g^m|_{Y_j}$ is the following composition of maps:

$$Y_j \xrightarrow{h} Y_{j+1} \xrightarrow{h} \dots \xrightarrow{h} Y_{m-1} \xrightarrow{\tau^{-1} \circ h} Y_0 \xrightarrow{h} \dots \xrightarrow{h} Y_j.$$

which thus coincides with $h^j \circ \tau^{-1} \circ h^{m-j} = \mathbf{F}_{\alpha_j} = \mathbf{F}_\beta$. $\square$

10. PROOF OF THEOREM 3.6

Let M be a compact surface, $f \in \mathcal{F}(M, P)$, $h \in \mathcal{S}(f)$, A be the union of all regular leaves of f being h^- -invariant and $K_1, \dots, K_k$ be all the critical leaves of f such that $\overline{A} \cap K_i \neq \emptyset$. For $i = 1, \dots, k$, let R_{K_i} be an f -regular neighborhood of K_i chosen so that $R_{K_i} \cap R_{K_j} = \emptyset$ for $i \neq j$ and

$$Z := A \bigcup \left(\bigcup_{i=1}^k R_{K_i} \right).$$

Assume that Z is non-empty, orientable and every connected component γ of $\partial Z \cap \text{Int } M$ separates M . We have to prove that there exists $g \in \mathcal{S}(f)$ which coincide with h on Z and such that $g^2 \in \mathcal{S}_{\text{id}}(f)$.

Lemma 10.1. *There exists a unique $\mathcal{C}^\infty$ function $\alpha : Z \rightarrow \mathbb{R}$ such that $h^2|_Z = \mathbf{F}_\alpha$ and $\alpha = 0$ at each fixed point of h^2 on h^- -invariant regular leaves of f .*

Proof. Let V be a regular leaf V of f , and $\mathcal{N}$ a chipped cylinder of f such that $V \subset \text{Int } \mathcal{N}$. If V is h^- -invariant, then so is every other regular leaf $V' \subset \text{Int } \mathcal{N}$. This implies that Z is a union of f -regular neighborhoods $Z_1, \dots, Z_l$ of some chipped cylinders $\mathcal{N}_1, \dots, \mathcal{N}_l$ of f .

By Theorem 8.3, for each $i = 1, \dots, l$ there exists a unique $\mathcal{C}^\infty$ function $\alpha_i : Z_i \rightarrow \mathbb{R}$ such that $h^2|_{Z_i} = \mathbf{F}_{\alpha_i}$ and $\alpha = 0$ at each fixed point of h^2 on each on h^- -invariant regular leaf of f in Z_i .

Notice that if $Z_i \cap Z_j \neq \emptyset$, then every connected component W of that intersection always contains a non-periodic orbit γ of $\mathbf{F}$. Therefore by uniqueness of shift functions (Corollary 4.5) we obtain that $\alpha_i = \alpha_j$ on W .

Hence the functions $\{\alpha_i\}_{i=1,\dots,l}$ agree on the corresponding intersections, and therefore they define a unique $\mathcal{C}^\infty$ function $\alpha : Z \rightarrow \mathbb{R}$ such that $h^2|_Z = \mathbf{F}_\alpha$. Then $\alpha = 0$ at each on h^- -invariant regular leaf of f in Z_i . $\square$

If $M = Z$, then theorem is proved. Thus suppose that $M \neq Z$.

Lemma 10.2. *The number of connected components $\overline{M \setminus Z}$ is even, and they can be enumerated by pairs of numbers:*

$$\begin{array}{cccc} Y_{1,0} & Y_{2,0} & \dots & Y_{a,0} \\ Y_{1,1} & Y_{2,1} & \dots & Y_{a,1} \end{array} \quad (10.1)$$

for some $a > 1$ so that h exchanges the rows in (10.1), that is $h(Y_{i,0}) = Y_{i,1}$ and $h(Y_{i,1}) = Y_{i,0}$ for each i .

Proof. Let $Y_1, \dots, Y_q$ be all the connected components of $\overline{M \setminus Z}$. Denote $\gamma_i := Y_i \cap Z$. Then by condition (B), γ_i is a unique common boundary component of Y_i and Z .

Since $h(Z) = Z$, it follows that h induces a permutation of connected components of $\{Y_i\}_{i=1,\dots,q}$. Moreover, by Lemma 3.5.1 $h(\gamma_i) \cap \gamma_i = \emptyset$, whence $h(Y_i) \cap Y_i = \emptyset$ as well. On the other hand, $h^2 = \mathbf{F}_\alpha$, whence $h^2(\gamma_i) = \mathbf{F}_\alpha(\gamma_i) = \gamma_i$, and therefore $h^2(Y_i) = Y_i$.

Thus $\{Y_i\}_{i=1,\dots,q}$ splits into pairs which are exchanged by h . $\square$

Now by Lemma 9.1 with $m = 2$ there exists g such that $g = h$ on Z and $g^2 \in \mathcal{S}_{\text{id}}(f)$. Theorem 3.6 is completed. $\square$

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