Reversing orientation homeomorphisms of surfaces

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To the memory of Volodymyr Vasylyovych Sharko

Abstract. Let \( M \) be a connected compact orientable surface, \( f : M \to \mathbb{R} \) be a Morse function, and \( h : M \to M \) be a diffeomorphism which preserves \( f \) in the sense that \( f \circ h = f \). We will show that if \( h \) leaves invariant each regular component of each level set of \( f \) and reverses its orientation, then \( h^2 \) is isotopic to the identity map of \( M \) via \( f \)-preserving isotopy.

This statement can be regarded as a foliated and a homotopy analogue of a well known observation that every reversing orientation orthogonal isomorphism of a plane has order 2, i.e. a mirror symmetry with respect to some line.

The obtained results hold in fact for a larger class of maps with isolated singularities from compact orientable surfaces to the real line and the circle.

Keywords: Diffeomorphism, Morse function, dihedral group

DOI: http://dx.doi.org/10.15673/tmgc.v13i4.1953
1. INTRODUCTION

The present paper describes several foliated and homotopy variants of a “rigidity” property for reversing orientation linear motions of the plane claiming that every such motion has order 2. Though it is motivated by study of deformations of smooth functions on surfaces (and we prove the corresponding statements), the obtained results seem to have an independent interest.

Let $D_n = \{r, s \mid r^n = s^2 = 1, rs = sr^{-1}\}$ be the dihedral group, i.e. group of symmetries of a right $n$-polygon. Then each “reversing orientation” element is written as $r^k s$ for some $k$ and has order 2:

\[(r^k s)^2 = r^k s r^k s = r^k s r^{-k} s = 1.\]

More generally, let $SO^{-}(2) = \left\{ \left( \begin{array}{cc} \sin t & \cos t \\ -\cos t & \sin t \end{array} \right) \mid t \in [0; 2\pi) \right\}$ be the set of reversing orientation orthogonal maps of $\mathbb{R}^2$, i.e. the adjacent class of $O(2)/SO(2)$ distinct from $SO(2)$. Then again one easily checks that each element of $SO^{-}(2)$ has order 2.

Another counterpart of this effect is that every motion of $\mathbb{R}$ which reverses orientation is given by the formula: $f(x) = a - x$ for some $a \in \mathbb{R}$, and therefore it has order 2, i.e. $f(f(x)) = x$.

Notice that such a property does not hold for “non-rigid motions”, like arbitrary homeomorphisms or diffeomorphisms of $\mathbb{R}$ or $S^1$. For example, let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(x) = -x^3$. It reverses orientation, but $h(h(x)) = -(-x^3)^3 = x^9$, and therefore it is not the identity.

Nevertheless, counterparts of the above rigidity effects for homeomorphisms can still be obtained on a “homotopy” level.

For instance, let $\mathcal{H}(S^1)$ be the group of all homeomorphisms of the circle $S^1$ and $\mathcal{H}^+(S^1)$, (resp. $\mathcal{H}^-(S^1)$), be the subgroup (resp. subset) consisting of homeomorphisms preserving (resp. reversing) orientation. Endow these spaces with compact open topologies. Notice that we have a natural inclusion $O(2) \subset \mathcal{H}(S^1)$, which consists of two inclusions $SO^{-}(2) \subset \mathcal{H}^-(S^1)$ and $SO(2) \subset \mathcal{H}^+(S^1)$ between the corresponding path components. It is well known ([3], see also Lemma 5.1 below) and is easy to show that $SO^{-}(2)$ (resp. $SO(2)$) is a strong deformation retract of $\mathcal{H}^-(S^1)$ (resp. $\mathcal{H}^+(S^1)$). This implies that the map $sq : \mathcal{H}^-(S^1) \to \mathcal{H}^+(S^1)$ defined by $sq(h) = h^2$ is null homotopic.

The aim of the present paper is to prove a parametric variant of the above “rigidity” statements for self-homeomorphisms of open subsets of topological products $X \times S^1$ preserving first coordinate (Theorem 6.2). That result will be applied to diffeomorphisms preserving a Morse function on an orientable surface and reversing certain regular components of some of its level-sets (Theorems 3.3, 3.5, and 3.6).
2. Preliminaries

2.1. Action of the groups of diffeomorphisms. Let $M$ be a compact surface and $P$ be either the real line $\mathbb{R}$ or the circle $S^1$ and $\mathcal{D}(M)$ be the group of $C^\infty$ diffeomorphisms of $M$. Then there is a natural right action of $\mathcal{D}(M)$ on the space of smooth maps $C^\infty(M, P)$ defined by the following rule: $(h, f) \mapsto f \circ h$, where $h \in \mathcal{D}(M)$, $f \in C^\infty(M, P)$. Let $S(f) = \{ h \in \mathcal{D}(M) \mid f \circ h = f \}$, $O(f) = \{ f \circ h \mid h \in \mathcal{D}(M) \}$, be the stabilizer and the orbit of $f \in C^\infty(M, P)$ with respect to the above action. It will be convenient to say that elements of $S(f)$ preserve $f$. Endow the spaces $\mathcal{D}(M)$, $C^\infty(M, P)$ with Whitney $C^\infty$-topologies, and their subspaces $S(f)$, $O(f)$ with induced ones. Denote by $S_{id}(f)$ the identity path component of $S(f)$ consisting of all $h \in S(f)$ isotopic to $id_M$ by some $f$-preserving isotopy.

For orientable $M$ we denote by $\mathcal{D}^+(M)$ the group of its orientation preserving diffeomorphisms. Also for $f \in C^\infty(M, P)$ we put

$$S^+(f) = \mathcal{D}^+(M) \cap S(f), \quad S^-(f) = S(f) \setminus S^+(f),$$

so $S^-(f)$ is another adjacent class of $S(f)$ by $S^+(f)$ distinct from $S^+(f)$.

2.2. Homogeneous polynomials on $\mathbb{R}^2$ without multiple factors. It is well known and is easy to prove that every homogeneous polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ is a product of finitely many linear and irreducible over $\mathbb{R}$ quadratic factors:

$$f = \prod_{i=1}^l L_i \cdot \prod_{j=1}^q Q_j,$$

where $L_i(x, y) = a_ix + b_iy$ and $Q_j(x, y) = c_ix^2 + d_iy + e_iy^2$.

Suppose $\deg f \geq 2$. Then the origin 0 is a unique critical point of $f$ if and only if $f$ has no multiple (non-proportional each other) linear factors. Structure of level sets of $f$ near 0 is shown in Figure 2.1.

![Figure 2.1. Topological structure of level sets of a homogeneous polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ without multiple factors](image)
We will restrict ourselves with the following space of smooth maps.

**Definition 2.2.1.** Let $\mathcal{F}(M, P)$ be the set of $C^\infty$ maps $f : M \to P$ satisfying the following conditions:

1. **(A1)** The map $f$ takes constant value at each connected component of $\partial M$ and has no critical points in $\partial M$.
2. **(A2)** For every critical point $z$ of $f$ there is a local presentation $f_z : \mathbb{R}^2 \to \mathbb{R}$ of $f$ near $z$ such that $f_z$ is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

A map $f \in C^\infty(M, P)$ will be called *Morse* if $f$ satisfies (A1) and all its critical points are non-degenerate. Notice that due to Morse lemma each Morse map $f$ satisfies the condition (A2) with homogeneous polynomials $f_z = \pm x^2 \pm y^2$ for each critical point $z$. Denote by $\text{Morse}(M, P)$ the set of all Morse maps $M \to P$. Then we have the following inclusion:

$$\text{Morse}(M, P) \subset \mathcal{F}(M, P) \subset C^\infty(M, P).$$

Then $\text{Morse}(M, P)$ is open and everywhere dense in the subset of smooth functions from $C^\infty(M, P)$ satisfying the condition (A2). Hence $\mathcal{F}(M, P)$ consists of *even more that typical maps* $M \to P$.

### 2.3. Several constructions associated with $f \in \mathcal{F}(M, P)$

In what follows we will assume that $f \in \mathcal{F}(M, P)$. Let $\Sigma_f$ be the set of critical points of $f$. Then condition (A2) implies that each $z \in \Sigma_f$ is isolated. In the case when $P = S^1$ one can also say about local extremes of $f$, and even about local minimums or maximums if we fix an orientation of $S^1$.

A connected component $K$ of a level-set $f^{-1}(c)$, $c \in P$, will be called a leaf (of $f$). We will call $K$ *regular* if it contains no critical points. Otherwise, it will be called *critical*.

For $\varepsilon > 0$ let $N_\varepsilon$ be the connected component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ containing $K$. Then $N_\varepsilon$ will be called an *$f$-regular neighborhood of $K$* if $\varepsilon$ is so small that $N_\varepsilon \setminus K$ contains no critical points of $f$ and no boundary components of $\partial M$.

A submanifold $U \subset M$ will be called *$f$-adapted* if $U = \bigcup_{i=1}^n A_i$, where each $A_i$ is either a regular leaf of $f$ or an $f$-regular neighborhood of some (regular or critical) leaf of $f$.

### 2.3.1. Graph of $f$.

Let $\Gamma_f$ be the partition of $M$ into leaves of $f$, and $p : M \to \Gamma_f$ be the natural map associating to each $x \in M$ the corresponding leaf of $f$ containing $x$. Endow $\Gamma_f$ with the quotient topology, so a subset (a collection of leaves) $U \subset \Gamma_f$ is open iff $p^{-1}(U)$ (i.e. their union) is open in $M$. It follows from axioms (A1) and (A2) that $\Gamma_f$ has a natural structure of 1-dimensional CW-complex, whose 0-cells correspond to
boundary components of \( M \) and critical leaves of \( f \). It is known as Reeb or Kronrod-Reeb or Lyapunov graph of \( f \), see [1, 2, 4, 10]. We will call \( \Gamma_f \) simply the graph of \( f \).

The following statement is evident:

**Lemma 2.3.2.** For \( f \in \mathcal{F}(M, P) \) the following conditions are equivalent:

(a) every regular leaf of \( f \) in \( \text{Int} \ M \) separates \( M \);
(b) the graph \( \Gamma_f \) of \( f \) is a tree.

For instance, those conditions hold if \( M \) is either of the surfaces: 2-disk, cylinder, 2-sphere, Möbius band, projective plane.

2.3.3. **Singular foliation of \( f \).** Consider finer partition \( \Xi_f \) of \( M \) whose elements are of the following three types:

(i) regular leaves of \( f \);
(ii) critical points of \( f \);
(iii) connected components of the sets \( K \setminus \Sigma_f \), where \( K \) runs over all critical leaves of \( f \) (evidently, each such component is an open arc).

In other words, each critical leaf of \( f \) is additionally partitioned by critical points. We will call \( \Xi_f \) the singular foliation of \( f \).

2.3.4. **Hamiltonian like flows of \( f \in \mathcal{F}(M, P) \).** Suppose \( M \) is orientable. A smooth vector field \( F \) on \( M \) will be called Hamiltonian-like for \( f \) if the following conditions hold true.

(a) \( F(z) = 0 \) if and only if \( z \) is a critical point of \( f \).
(b) \( df(F) \equiv 0 \) everywhere on \( M \), i.e. \( f \) is constant along orbits of \( F \).
(c) Let \( z \) be a critical point of \( f \). Then there exists a local representation of \( f \) at \( z \) in the form of a homogeneous polynomial \( g : \mathbb{R}^2 \to \mathbb{R} \) without multiple factors as in Axiom (A2), such that in the same coordinates \( (x, y) \) near the origin 0 in \( \mathbb{R}^2 \) we have that \( F = -g'_y \frac{\partial}{\partial x} + g'_x \frac{\partial}{\partial y} \).

It follows from (a) and Axiom (A1) that the orbits of \( F \) precisely elements of the singular foliation \( \Xi_f \).

By [7, Lemma 5.1] or [8, Lemma 16] every \( f \in \mathcal{F}(M, P) \) admits a Hamiltonian-like vector field.

The following statement is a principal technical result established in a series of papers by the second author, see [9, Lemma 6.1(iv)] for details:

**Lemma 2.3.5.** Let \( F : M \times \mathbb{R} \to M \) be a Hamiltonian like flow for \( f \) and \( h \in \mathcal{S}(f) \). Then \( h \in \mathcal{S}_{\text{id}}(f) \) if and only if there exists a \( C^\infty \) function \( \alpha : M \to \mathbb{R} \) such that \( h(x) = F(x, \alpha(x)) \) for all \( x \in M \). Moreover, in this case the homotopy \( H : M \times [0; 1] \to M \) given by \( H(x, t) = F(x, t\alpha(x)) \) is an isotopy between \( H_0 = \text{id}_M \) and \( H_1 = h \) in \( \mathcal{S}(f) \).
3. MAIN RESULTS

Let \( M \) be a compact surface, \( h : M \to M \) a homeomorphism, and \( \gamma \subset M \) a submanifold of \( M \). Then \( \gamma \) is \( h \)-invariant, whenever \( h(\gamma) = \gamma \).

Moreover, suppose \( \gamma \) is connected and orientable \( h \)-invariant submanifold with \( \dim \gamma \geq 1 \). Then \( \gamma \) is \( h^+ \)-invariant (resp. \( h^- \)-invariant) whenever the restriction \( h|_\gamma : \gamma \to \gamma \) preserves (resp. reverses) orientation of \( \gamma \). If \( \gamma \) is a fixed point of \( h \), then we assume that \( \gamma \) is mutually \( h^+ \)- and \( h^- \)-invariant.

We will be interesting in the structure of diffeomorphisms preserving \( f : \mathcal{F}(M, p) \) and reversing orientations of some regular leaves of \( f \). The following easy lemma can be proved similarly to [7, Lemma 3.5].

**Lemma 3.1.** Let \( M \) be a connected orientable surface and \( h \in \mathcal{S}(f) \) be such that every regular leaf of \( f \) is \( h \)-invariant. Then every critical leaf of \( f \) in also \( h \)-invariant and the following conditions are equivalent:

1. some regular leaf of \( f \) is \( h^+ \)-invariant;
2. all regular leaves of \( f \) are \( h^+ \)-invariant;
3. \( h \) preserves orientation of \( M \).

**Counterexample 3.2.** Lemma 3.1 fails for non-orientable surfaces. Let \( M \) be a Möbius band, and \( f : M \to \mathbb{R} \) be a Morse function having two critical points: one local extreme \( z \) and one saddle \( y \). Let \( K \) be the critical leaf of \( f \) containing \( y \), and \( D \) and \( E \) be the connected component of \( M \setminus K \) containing \( z \) and \( \partial M \) respectively. Then it is easy to construct a diffeomorphism \( h : M \to M \) (called slice along central circle of Möbius band) which is fixed on \( \partial M \), and for which \( D \) is \( h^- \)-invariant. Then regular leaves of \( f \) in \( D \) are \( h^- \)-invariant, while regular leaves of \( f \) in \( E \) are \( h^+ \)-invariant.

Denote by \( \Delta^-(f) \) the subset of \( \mathcal{S}(f) \) consisting of diffeomorphisms \( h \) such that every regular leaf of \( f \) is \( h^- \)-invariant.

Let \( h \in \Delta^-(f) \). Then by Lemma 3.1 every critical leaf \( K \) of \( f \) is \( h^- \)-invariant, however, \( h \) may interchange critical points of \( f \) in \( K \) and the leaves of \( \Xi_f \) contained in \( K \) (i.e. connected components of \( K \setminus \Sigma_f \)). Notice also that in general \( \Delta^-(f) \) can be empty.

The following theorem can be regarded as a homotopical and foliated variant of the above rigidity property for diffeomorphisms preserving \( f \).

**Theorem 3.3.** Let \( M \) be a compact orientable surface, \( f \in \mathcal{F}(M, p) \), and \( h \in \Delta^-(f) \). Then \( h^2 \in \mathcal{S}_{\text{id}}(f) \). In particular, for every \( h \in \Delta^-(f) \) each leaf of \( \Xi_f \) is \( (h^2, +) \)-invariant.

**Corollary 3.4.** Let \( \text{or} : \pi_0 \mathcal{S}(f) \to \mathbb{Z}_2 = \{0, 1\} \) be the orientation homomorphism defined by \( \text{or}(\mathcal{S}^+(f)) = 0 \) and \( \text{or}(\mathcal{S}^-(f)) = 1 \). Then for each
h \in \Delta^-(f) \text{ the map } s : \mathbb{Z}_2 \to \pi_0 S(f), \text{ defined by } s(0) = [\text{id}_M] \text{ and } s(1) = [h] \text{ is a homomorphism satisfying } \text{or} \circ s = \text{id}_{\mathbb{Z}_2}. \text{ In other words, } s \text{ is a section of } \text{or}, \text{ whence } \pi_0 S(f) \text{ is a certain semidirect product of } \pi_0 S^+(f) \text{ and } \mathbb{Z}_2.

**Proof.** One should just mention that Theorem 3.3 implies that \([h]^2 = [\text{id}_M]\) in \(\pi_0 S(f)\). □

Now consider the situation, when not all regular leaves are \(h^\sim\)-invariant. To clarify the situation we first formulate a particular case of the general statement for maps on 2-disk and cylinder.

**Theorem 3.5.** Let \(M\) be a compact orientable surface, \(f \in \mathcal{F}(M, P)\), \(V\) be a regular leaf of \(f\), and \(h \in S(f)\). Suppose every regular leaf of \(f\) in Int \(M\) separates \(M\) (this holds e.g. when \(M\) is a 2-disk, cylinder or a 2-sphere, see Lemma 2.3.2), and that \(V\) is \(h^\sim\)-invariant. Then there exists \(g \in S(f)\) which coincide with \(h\) on some neighborhood of \(V\) and such that \(g^2 \in S_{\text{id}}(f)\).

Emphasize that Theorem 3.5 does not claim that \(g \in \Delta^-(f)\), though \(g\) reverses orientation of \(V\) and its square belong to \(S_{\text{id}}(f)\).

**General result.** Theorems 3.3 and 3.5 are consequences of the following Theorem 3.6 below. Let \(M\) be a compact (not necessarily orientable) surface, \(f \in \mathcal{F}(M, P)\), and \(h \in S(f)\). Let also

- \(A\) be the union of all \(h^\sim\)-invariant regular leaves of \(f\),
- \(K_1, \ldots, K_k\) be all the critical leaves of \(f\) such that \(\overline{A} \cap K_i \neq \emptyset\);
- for \(i = 1, \ldots, k\), let \(R_{K_i}\) be an \(f\)-regular neighborhood of \(K_i\) chosen so that \(R_{K_i} \cap R_{K_j} = \emptyset\) for \(i \neq j\) and

\[
Z := A \bigcup \left( \bigcup_{i=1}^{k} R_{K_i} \right).
\]

Evidently, \(Z\) is an \(f\)-adapted subsurface of \(M\) and each of its connected components intersects \(A\).

**Lemma 3.5.1.** Suppose \(Z\) is orientable. Let also \(\gamma\) be a boundary component of \(Z\). Consider the following conditions:

1. \(\gamma \subset \text{Int } M\);
2. \(\gamma = \partial U \cap \partial Z\) for some connected component \(U\) of \(M \setminus Z\);
3. \(h(\gamma) \cap \gamma = \emptyset\);
4. \(h(\gamma) \neq \gamma\).

Then (1) \(\iff\) (2) \(\Rightarrow\) (3) \(\iff\) (4).

**Proof.** (1) \(\iff\) (2) is evident, and (3) \(\iff\) (4) follows from the observation that \(\gamma\) is a regular leaf of \(h\) and \(h\) permutes leaves of \(f\).
(2)$\Rightarrow$(4) Suppose $h(\gamma) = \gamma$. Let $Z'$ be a connected component of $Z$ containing $\gamma$. Then by the construction $Z'$ must intersect $A$, whence $h$ changes orientation of some regular leaves in $Z'$. Since $Z'$ is also orientable, we get from Lemma 3.1 that $h$ also changes orientation of $\gamma$. This means that $\gamma \subset A$, and therefore there exists an open neighborhood $W \subset A$ of $\gamma$ consisting of regular leaves of $f$. In particular, the regular leaves in $W \cap \text{Int} U$ must be contained in $A$ which contradicts to the assumption that $\text{Int} U \subset M \setminus Z \subset M \setminus A$.

Thus $h$ interchanges boundary components of $Z$ belonging to the interior of $M$. We will introduce the following property on $Z$:

(B) every connected component of $\partial Z \cap \text{Int} M$ separates $M$.

This condition means that there is a bijection between boundary components of $\partial Z \cap \text{Int} M$ and connected components of $M \setminus Z$, whence by Lemma 3.5.1 there will be no $h$-invariant connected components of $M \setminus Z$.

**Theorem 3.6.** If $Z$ is non-empty, orientable and has property (B), then there exists $g \in \mathcal{S}(f)$ such that $g = h$ on $Z$ and $g^2 \in \mathcal{S}_{\text{id}}(f)$.

**Proof of Theorem 3.3.** Suppose $M$ is orientable and $h \in \Delta^-(f)$. Then in the notation of Theorem 3.6, $Z = M$, and by that theorem there exists $g \in \mathcal{S}(f)$ such that $g = h$ on $Z$ and $g^2 \in \mathcal{S}_{\text{id}}(f)$. This means that $h = g$ and $h^2 \in \mathcal{S}_{\text{id}}(f)$.

**Proof of Theorem 3.5.** The assumption that every regular leaf of $f$ in $\text{Int} M$ separates $M$ implies condition (B).

3.7. **Structure of the paper.** In Section 4 we discuss a notion of a shift map along orbits of a flow which was studied in a series of papers by the second author, and extend several results to continuous flows. Section 5 devoted to reversing orientation families of homeomorphisms of the circle. In Section 6 we study flows without fixed point, and in Section 7 recall several results about passing from a flow on the plane to the flow written in polar coordinates. Section 8 introduces a certain subsurfaces of a surface $M$ associated with a map $f \in \mathcal{F}(M, P)$ and called *chipped cylinders*. We prove Theorem 8.3 describing behavior of diffeomorphisms reversing regular leaves of $f$ contained in those chipped cylinders. In Section 9 we prove Lemma 9.1 allowing to change $f$-preserving diffeomorphisms so that its finite power will be isotopic to the identity by $f$-preserving isotopy. Finally, in sections 9 and 10 we prove Theorem 3.6.
4. Shifts along orbits of flows

In this section we extend several results obtained in [5, 9] for smooth flows to a continuous situation. Let $X$ be a topological space.

**Definition 4.1.** A continuous map $F : X \times \mathbb{R} \to X$ is a *(global) flow on* $X$, if $F_0 = \text{id}_X$ and $F_s \circ F_t = F_{s+t}$ for all $s, t \in \mathbb{R}$, where $F_s : X \to X$ is given by $F_s(x) = F(x, s)$. For $x \in X$ the subset $F(x \times \mathbb{R}) \subset X$ is called the *orbit* of $x$.

It is well known that a $C^r$, $1 \leq r \leq \infty$, vector field $F$ of a smooth compact manifold $X$ tangent to $\partial X$ always generates a flow $F$.

Assume further that $F$ is a flow on a topological space $X$. Let also $V \subset X$ be a subset. Say that a continuous map $h : V \to X$ preserves orbits of $F$ on $V$ if $h(\gamma \cap V) \subset \gamma$ for every orbit $\gamma$ of $F$. The latter means that for each $x \in U$ there exists a number $\alpha_x \in \mathbb{R}$ such that $h(x) = F(x, \alpha_x)$. Notice that in general $\alpha_x$ is not unique and does not continuously depend on $x$.

Conversely, let $\alpha : V \to \mathbb{R}$ be a continuous function such that its graph $\Gamma_\alpha = \{(x, \alpha(x)) \mid x \in V\}$ is contained in $W$. For a global flow any continuous $\alpha : V \to \mathbb{R}$ satisfies that condition. Then one can define the following map $F_\alpha : V \to X$ by

$$F_\alpha(z) := F(z, \alpha(z)) = F_{\alpha(z)}(z).$$

We will call $F_\alpha$ a *shift along orbits of $F$ by the function $\alpha$*, while $\alpha$ will be called a *shift function* for $F_\alpha$.

Notice that $F_\alpha$ preserves orbits of $F$ on $V$, and in general is not a homeomorphism.

**Definition 4.2.** Let $x$ be a non-fixed point of $F$, and $Y$ be a topological space. Let also $U$ be an open neighborhood of $x$, and $\phi = (\zeta, \rho) : U \to Y \times \mathbb{R}$ be an open embedding. Then the pair $(\phi, U)$ will be called a *flow-box chart* at $x$, if there exist an $\varepsilon > 0$ and an open neighborhood $V$ of $x$ in $X$ such that

$$\phi \circ F(z, t) = (\zeta(z), \rho(z) + t)$$

for all $(z, t) \in V \times (-\varepsilon; \varepsilon)$.

In other words, $F$ is *locally conjugated* to the flow $G : (Y \times \mathbb{R}) \times \mathbb{R} \to Y \times \mathbb{R}$ defined by $G(y, s, t) = (y, s + t)$, since the identity (4.1) can be written as

$$\phi \circ F(z, t) = (\zeta(z), \rho(z) + t) = G(\zeta(z), \rho(z), t) = G(\phi(z), t),$$

i.e.

$$\phi \circ F_t(z) = G_t \circ \phi(z).$$

(4.2)

It is well known that each $C^2$ flow (generated by some $C^1$ vector field) on a smooth manifold admits flow-box charts at each non-fixed point.
Lemma 4.3. If \((\phi, U)\) is a flow box chart at a non-fixed point \(x \in X\) of \(F\), then for any \(\tau \in \mathbb{R}\) the pair \((\phi \circ F_{-\tau}, F_{\tau}(U))\) is a flow box chart at the point \(y = F(x, \tau)\).

Proof. Denote \(U' = F_{\tau}(U)\) and \(\phi' = \phi \circ F_{-\tau}\). Let also \(\varepsilon\) and \(V\) be as in Definition 4.1, and \(V' = F_{\tau}(V)\) be a neighborhood of \(y\). Then for any \((z, t) \in V' \times (-\varepsilon, \varepsilon)\) we have that

\[
\phi'(z) = (\phi \circ F_{-\tau})(z) = (\phi \circ F_{t-\tau})(z) = (\phi \circ F_t(F_{-\tau}(z))) = G_t \circ \phi(F_{-\tau}(z)) = G_t \circ \phi'(z).
\]

The following lemma shows that for flows admitting flow box charts (e.g. for smooth flows) every orbit preserving map admits a shift function near each non-fixed point. Moreover, such a function is locally determined by its value at that point.

Lemma 4.4. (cf. [9, Lemma 6.1(i)]). Suppose a flow \(F : X \times \mathbb{R} \to X\) has a flow-box \((\phi, U)\) at some non-fixed point \(x\). Let also \(h : U \to X\) be a continuous map preserving orbits of \(F\) and such that \(h(x) = F(x, \tau)\) for some \(\tau \in \mathbb{R}\). Then there exists an open neighborhood \(V \subset U\) of \(x\) and a unique continuous function \(\alpha : V \to \mathbb{R}\) such that

1. \(\alpha(x) = \tau\);
2. \(h(z) = F(z, \alpha(z))\) for all \(z \in V\).

If in addition \(X\) is a manifold of class \(C^r\), \((1 \leq r \leq \infty)\), \(F\) and \(h\) are \(C^r\), and \(\phi\) is a \(C^r\) embedding, then \(\alpha\) is \(C^r\) as well.

Proof. The proof almost literally repeats the arguments of [9, Lemma 6.2] proved for smooth flows and based on existence of flow box charts. For completeness we present a short proof for continuous situation.

1) First suppose \(\tau = 0\), so \(h(x) = x\). Let \(V\) and \(\varepsilon\) be the same as in Definition 4.2. Decreasing \(V\) one can also assume that \(V \subset U \cap h^{-1}(U)\), so in particular \(h(V) \subset U\). Denote \(\hat{V} := \phi(V)\) and \(\hat{U} := \phi(U)\). Then these sets are open, and we have a well-defined map \(\hat{h} = \phi \circ h \circ \phi^{-1} : \hat{V} \to \hat{U}\) which preserves orbits of \(\hat{G}\) due to (4.2). This means that \(\hat{h}(y, t) = (y, \eta(y, t))\) for some continuous function \(Y \times \mathbb{R} \ni (y, t) \mapsto \eta(y, t) \in \mathbb{R}\). Define another continuous function \(\alpha' : \hat{V} \to \mathbb{R}\) by \(\alpha'(y, t) = \eta(y, t) - t\). Then

\[
\hat{h}(y, t) = (y, t + \alpha'(y, t)) = G(y, t, \alpha'(y, t)) = G_{\alpha'}(y, t),
\]

that is \(\phi \circ h \circ \phi^{-1} = \hat{h} = G_{\alpha'}\), whence for each \(z \in V\) we have that

\[
h(z) = \phi^{-1} \circ G_{\alpha'} \circ \phi(z) = \phi^{-1} \circ G_{\alpha' \circ \phi(z)} \circ \phi(z) = F_{\alpha' \circ \phi(z)}(z) = F_{\alpha' \circ \phi(z)}.
\]
Thus one can put $\alpha = \alpha' \circ \phi : V \to \mathbb{R}$. It follows that $\alpha$ is continuous. Moreover, let $\hat{\eta}, \hat{\tau} = \phi(x)$. Since $h(x) = x$, we get that $\hat{h}(\hat{\eta}, \hat{\tau}) = \hat{\eta}, \hat{\tau}$, whence $\eta(\hat{\eta}, \hat{\tau}) = \hat{\tau}$, whence $\alpha'(\hat{\eta}, \hat{\tau}) = 0$, and thus

$$\alpha(x) = \alpha' \circ \phi(x) = \alpha'(\hat{\eta}, \hat{\tau}) = 0.$$  

2) If $\tau \not= 0$, then consider the map $\tilde{h} : U \to X$ given by $\tilde{h} = F_{\tau} \circ h$. Then $\tilde{h}(x) = F_{\tau} \circ h(x) = F(F(x, \tau), -\tau) = x$, whence by 1) $\tilde{h} = F_{\hat{\alpha}}$ for a unique continuous function $\hat{\alpha} : V \to \mathbb{R}$ such that $\hat{\alpha}(x) = 0$. Put $\alpha = \hat{\alpha} + \tau$. Then $\alpha(x) = \tau$ and

$$h(z) = F_{\tau} \circ \tilde{h}(z) = F(F(z, \hat{\alpha}(z)), \tau) = F(z, \hat{\alpha}(z) + \tau) = F(z, \alpha(z)). \quad \square$$

The formulas for $\alpha$ imply that if $X$, $F$, $\phi$, and $h$ are $C^r$, $(1 \leq r \leq \infty)$, then $\alpha$ is $C^r$ as well.

**Corollary 4.5.** Suppose $U \subset X$ is an open connected subset such that every $x \in U$ is non-fixed and admits a flow-box. Let also $\alpha, \alpha' : U \to \mathbb{R}$ be two continuous functions such that $F_{\alpha} = F_{\alpha'}$ on $U$. If $\alpha(x) = \alpha'(x)$ at some $x \in U$ (this holds e.g. if $F$ has at least one non-periodic point in $U$), then $\alpha = \alpha'$ on $U$.

**Proof.** The set $A = \{\alpha(y) = \alpha'(y) \mid y \in U\}$ is closed and by assumption non-empty (contains $x$). Moreover, by Lemma 4.4 this set is open, whence $A = U$. \quad \square

**Corollary 4.6.** Let $F : X \times \mathbb{R} \to X$ be a continuous flow, $U \subset X$ be an open subset such that every point $x \in U$ is non-fixed and non-periodic and admits a flow box chart, and $h : U \to X$ be an orbit preserving map. Then there exists a unique continuous function $\alpha : U \to \mathbb{R}$ such that $h(x) = F(x, \alpha(x))$ for all $x \in U$.

If in addition $X$ is a manifold of class $C^r$, $(0 \leq r \leq \infty)$, $F$ is $C^r$ and admits $C^r$ flow box charts, and $h$ is $C^r$, then $\alpha$ is $C^r$ as well.

**Proof.** Since every point $x \in U$ is non-fixed and non-periodic, there exists a unique number $\alpha(x)$ such that $h(x) = F(x, \alpha(x))$. Moreover, since $F$ admits flow box chart at $x$, it follows from Lemma 4.4 that the correspondence $x \mapsto \alpha(x)$ is a continuous function $\alpha : U \to \mathbb{R}$, which is also $C^r$ under the corresponding smoothness assumptions on $X$, $F$, $\phi$, and $h$. \quad \square

The following statement extends some results established in [9] for smooth flows to continuous flows having flow box charts at each non-fixed point on arbitrary topological spaces.
Lemma 4.7. (c.f. [9, Lemmas 6.1-6.3]) Let $F : X \times \mathbb{R} \to X$ be a flow, $p : \tilde{X} \to X$ a covering map, and $\xi : \tilde{X} \to \tilde{X}$ a covering transformation, i.e. a homeomorphism such that $p \circ \xi = p$. Then the following statements hold.

1. $F$ lifts to a unique flow $\tilde{F} : \tilde{X} \times \mathbb{R} \to \tilde{X}$ such that $p \circ \tilde{F}_t = F_t \circ p$ for all $t \in \mathbb{R}$.

2. For each continuous function $\alpha : X \to \mathbb{R}$ the map $\tilde{F}_{\alpha \circ p} : \tilde{X} \to \tilde{X}$ is a lifting of $F_\alpha$, that is

$$\tilde{F}_{\alpha \circ p} \circ p = F_\alpha \circ p.$$

3. $\tilde{F}$ commutes with $\xi$ in the sense that $\tilde{F}_t \circ \xi = \xi \circ \tilde{F}_t$ for all $t \in \mathbb{R}$.

More generally, for any function $\beta : \tilde{X} \to \mathbb{R}$ we have $\tilde{F}_\beta \circ \xi = \xi \circ \tilde{F}_\beta \circ \xi$.

4. For a continuous function $\beta : \tilde{X} \to \mathbb{R}$ and a point $z \in \tilde{X}$ consider the following (“global” and “point”) conditions:

\[
\begin{align*}
(g1) \; & \beta = \beta \circ \xi; \\
(g2) \; & \tilde{F}_\beta = \tilde{F}_{\beta \circ \xi}; \\
(g3) \; & \tilde{F}_\beta \circ \xi = \xi \circ \tilde{F}_\beta; \\
(p1) \; & \beta(z) = \beta \circ \xi(z); \\
(p2) \; & \tilde{F}_\beta(z) = \tilde{F}_{\beta \circ \xi}(z); \\
(p3) \; & \tilde{F}_\beta \circ \xi(z) = \xi \circ \tilde{F}_\beta(z).
\end{align*}
\]

Then we have the following diagram of implications:

\[
\begin{array}{ccc}
(g1) & \iff & (g2) \\
\downarrow & & \downarrow \\
(p1) & \iff & (p2) \\
\downarrow & & \downarrow \\
(p3) & \iff & (p3) \\
\end{array}
\]

(i) If $z$ is non-fixed and non-periodic point for $\tilde{F}$, then $(p2) \Rightarrow (p1)$.

(ii) If $\tilde{X}$ is path connected and $\tilde{F}_\beta$ is a lifting of some continuous map $h : X \to X$, that is $h \circ p = p \circ \tilde{F}_\beta$, then $(p3) \Rightarrow (g3)$, whence all conditions in the right square of (4.3) are equivalent.

(iii) If $\tilde{X}$ is path connected, and $\tilde{F}$ has no fixed points and admits flow box charts at each point of $\tilde{X}$, then $(g2) \& (p1) \Rightarrow (g1)$.

Proof. (1) Consider the following homotopy (with “open ends”)

$$G = F \circ (p \times \text{id}_\mathbb{R}) : \tilde{X} \times \mathbb{R} \to X, \quad G(z, t) = F(p(z), t),$$

and let $\tilde{F} : \tilde{X} \times 0 \to \tilde{X}$ be given by $\tilde{F}(z, 0) = z$. Then $\tilde{F}$ is a lifting of $G|_{\tilde{X} \times 0}$, that is $p \circ \tilde{F}(z, 0) = p(z) = F(p(z), 0) = G(z, 0)$. Hence $\tilde{F}$ extends to a unique lifting $\tilde{F} : \tilde{X} \times \mathbb{R} \to \tilde{X}$ of $G$. One easily checks that this lifting is a flow on $\tilde{X}$.

(2) Let $z \in \tilde{X}$ and $t = \alpha \circ p(z)$. Then

\[
p \circ \tilde{F}_{\alpha \circ p(z)}(z) = p \circ \tilde{F}_t(z) = F_t \circ p(z) = F_{\alpha \circ p(z)} \circ p(z) = F_\alpha \circ p(z).
\]
(3) Notice that the map $\tilde{F}' : \tilde{X} \times \mathbb{R} \to \tilde{X}$ defined by $\tilde{F}'_t = \xi \circ \tilde{F}_t \circ \xi^{-1}$ is also a flow on $\tilde{X}$. Moreover,

$$p \circ \tilde{F}'_t = p \circ \xi \circ \tilde{F}'_t \circ \xi^{-1} = p \circ \tilde{F}_t \circ \xi^{-1} = \tilde{F}_t \circ p \circ \xi^{-1} = \tilde{F}_t \circ p.$$ 

Thus $\tilde{F}'$ and $\tilde{F}$ are two liftings of $F$ which coincide at $t = 0$, and therefore $\tilde{F}' = \tilde{F}$ by uniqueness of liftings. Therefore for any continuous function $\tilde{\alpha} : \tilde{X} \to \mathbb{R}$ such that $\tilde{h} = \tilde{F}_0 \circ \tilde{\alpha}$, let

$$\tilde{F}_\tilde{\alpha} \circ \xi(z) = \tilde{F}(\xi(z), \tilde{\alpha}(\xi(z))) = \tilde{F}_{\tilde{\alpha}(\xi(z))} \circ \xi(z) = \xi \circ \tilde{F}_{\tilde{\alpha} \circ \xi(z)}(z) = \xi \circ \tilde{F}_{\tilde{\alpha} \circ \xi}(z).$$

(4) The implication in Diagram (4.3) are trivial. Assume that $\tilde{X}$ (and therefore $X$) are path connected.

(i) If $z$ is non-fixed and non-periodic, then $\tilde{F}(z, a) = \tilde{F}(z, b)$ implies that $a = b$ for any $a, b$. In particular, this hold for $a = \beta(z)$ and $b = \beta \circ \xi(z)$.

(ii) Suppose that $\tilde{F}_\beta$ is a lifting of some continuous map $h : X \to X$. Then $p \circ \tilde{F}_\beta \circ \xi = h \circ p \circ \xi = h \circ \tilde{p}$ and $p \circ \xi \circ \tilde{F}_\beta = p \circ \tilde{F}_\beta = h \circ p$, i.e. both $\tilde{F}_\beta \circ \xi$ and $\xi \circ \tilde{F}_\beta$ are liftings of $h$.

Now if (p3) holds, i.e. $\tilde{F}_\beta \circ \xi(z) = \xi \circ \tilde{F}_\beta(z)$ at some point $z \in \tilde{X}$, then these liftings must coincide on all of $\tilde{X}$, which means condition (g3).

(iii) Suppose $\tilde{F}$ has no fixed points and conditions (g2) and (p1) hold, that is $\tilde{F}(y, \beta(y)) = \tilde{F}(y, \beta \circ \xi(y))$ for all $y \in \tilde{X}$ and $\beta(z) = \beta \circ \xi(z)$ for some $z \in \tilde{X}$. Notice that the set $A = \{y \in \tilde{X} \mid \beta(y) = \beta \circ \xi(y)\}$ is close. Moreover, by the local uniqueness of shift-functions (Corollary 4.5) $A$ is also open. Since $\tilde{X}$ is connected, $A$ is either $\emptyset$ or $\tilde{X}$. But $z \in A$, whence $A = \tilde{X}$, i.e. condition (g1) holds. □

Let $p : \tilde{X} \to X$ be a regular covering map with path connected $\tilde{X}$ and $X$, $G$ be the group of covering transformation, $F : X \times \mathbb{R} \to X$ be a flow on $X$, and $\tilde{F} : \tilde{X} \times \mathbb{R} \to \tilde{X}$ be its lifting as in Lemma 4.7(1).

**Corollary 4.8.** (c.f. [9, Lemma 6.3]) Suppose all orbits of $\tilde{F}$ are non-fixed and non-closed and $\tilde{F}$ has flow box charts at all points of $\tilde{X}$. Let also $h : X \to X$ be a continuous map admitting a lifting $\tilde{h} : \tilde{X} \to \tilde{X}$, i.e. $p \circ \tilde{h} = h \circ p$, such that $\tilde{h}$ leaves invariant each orbit $\gamma$ of $\tilde{F}$ and commutes with each $\xi \in G$. Then there exist a unique continuous function $\alpha : X \to \mathbb{R}$ such that $h = F_\alpha$.

Again if $\tilde{X}$, $X$, $p$, $h$, $F$ and its flow box charts are $\mathcal{C}^r$, $1 \leq r \leq \infty$, then $\alpha$ is also $\mathcal{C}^r$.

**Proof.** Due to assumptions on $\tilde{F}$ we get from Corollary 4.6 that there exists a unique continuous function $\beta : \tilde{X} \to \mathbb{R}$ such that $\tilde{h} = \tilde{F}_\beta$. Let
\[\xi \in G.\] Since \(\tilde{h}\) commutes with \(\xi\), i.e. condition (g3) of Lemma 4.7 holds, we obtain the following implications:

\[
(g3) \iff (g2) \implies (p2) \implies (p1) \implies (g1).
\]

meaning that \(\beta \circ \xi = \beta\). Thus \(\beta\) is invariant with respect to all \(\xi \in G\), and therefore it induces a unique function \(\alpha : X \to \mathbb{R}\) such that \(\beta = \alpha \circ p\). Since \(p\) is a local homeomorphism, it follows that \(\alpha\) is continuous. Moreover, by Lemma 4.7(2), \(\tilde{h} = \tilde{F}_\beta = \tilde{F}_{\alpha \circ p}\) is a lifting of \(F_\alpha\). But \(\tilde{h}\) is also a lifting of \(h\), whence \(h = F_\alpha\).

Statements about smoothness of \(\alpha\) follows from the corresponding smoothness parts of used lemmas. We leave the details for the reader. \(\square\)

5. SELF MAPS OF THE CIRCLE

Let \(S^1 = \{z \in \mathbb{C} \mid |z| = 1\}\) be the unit circle in the complex plane, \(p : \mathbb{R} \to S^1\) be the universal covering map defined by \(p(s) = e^{2\pi is}\) and \(\xi(s) = s + 1\) be a diffeomorphism of \(\mathbb{R}\) generating the group of covering slices \(\mathbb{Z}\).

Denote by \(C_k(S^1, S^1)\), \(k \in \mathbb{Z}\), the set of all continuous maps \(h : S^1 \to S^1\) of degree \(k\), i.e. maps homotopic to the map \(z \mapsto z^k\). Then \(\{C_k(S^1, S^1)\}_{k \in \mathbb{Z}}\) is a collection of all path components of \(\mathcal{C}(S^1, S^1)\) with respect to the compact open topology.

For a map \(h : X \to X\) it will be convenient to denote the composition \(h \circ \cdots \circ h\) by \(h^n\) for \(n \in \mathbb{N}\). A point \(x \in X\) is fixed for \(h\), whenever \(h(x) = x\).

The following lemma collects simple properties of continuous maps of the circle which are usually referred as exercises. We present

**Lemma 5.1.** 1) Let \(h : S^1 \to S^1\) be a continuous map and \(\tilde{h}_0 : \mathbb{R} \to \mathbb{R}\) be any lifting of \(h\) with respect to \(p\), i.e. a continuous map making commutative the following diagram:

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\tilde{h}_0} & \mathbb{R} \\
p \downarrow & & \downarrow p \\
S^1 & \xrightarrow{h} & S^1
\end{array}
\]

that is \(p \circ \tilde{h}_0 = h \circ p\), or \(e^{2\pi i \tilde{h}_0(s)} = h(e^{2\pi is})\) for \(s \in \mathbb{R}\). For \(a \in \mathbb{Z}\) define the map \(\tilde{h}_a : \mathbb{R} \to \mathbb{R}\) by \(\tilde{h}_a = \xi^a \circ \tilde{h}_0\), that is \(\tilde{h}_a(s) = \tilde{h}_0(s) + a\). Then the following conditions are equivalent:

(a) \(h \in C_k(S^1, S^1)\);

(b) \(\tilde{h}_0 \circ \xi = \xi^k \circ \tilde{h} = \tilde{h}_k\), that is \(\tilde{h}(s + 1) = \tilde{h}(s) + k\) for all \(s \in \mathbb{R}\).
2) Define the map \( \phi_k : S^1 \to C_k(S^1, S^1) \), \( \phi_k(w)(z) = wz^k \), \( w, z \in S^1 \), is a homotopy equivalence. Moreover, the inclusions \( \phi_1 : SO(2) \subset H^+(S^1) \) and \( \phi_{-1} : SO^-(2) \subset H^-(S^1) \) are homotopy equivalences as well.

3) Let \( h \in C_k(S^1, S^1) \). Then

(i) \( h \) has at least \( |k - 1| \) fixed points;
(ii) \( \{ \tilde{h}_{ak} \}_{a \in \mathbb{Z}} \) is the collection of all possible liftings of \( h \);
(iii) \( \tilde{h}_a \circ \tilde{h}_b = \xi^{a+kb} \circ h^2 \) for any \( a, b \in \mathbb{Z} \);
(iv) if \( k = -1 \), then \( \tilde{h}_a^2 = \tilde{h}_0^2 \) for all \( a \in \mathbb{Z} \), in other words, for any lifting \( \tilde{h}_a \) of \( h_0 \) its square \( \tilde{h}_a^2 \) does not depend on \( a \). Moreover, if \( A \) is the set of fixed points of \( h^2 \), then \( p^{-1}(A) \) is the set of fixed points of \( \tilde{h}_0^2 \).

**Proof.** Statement 1) is easy.

2) For \( k = \pm 1 \) this statement was initially proved by H. Kneser, see last paragraph on page 367 of [3]. Notice that

\[
\phi_k \left( e^{2\pi is} \right) \left( e^{2\pi ix} \right) = e^{2\pi i(s+kx)}.
\]

for all \( s, x \in \mathbb{R} \). In particular, \( \phi_1 \) is the inclusion \( SO(2) \subset H^+(S^1) \), and \( \phi_{-1} \) is the inclusion \( SO^-(2) \subset H^-(S^1) \). We will show that the image of \( \phi_k \) is a strong deformation retract of \( C_k(S^1, S^1) \).

Let \( h \in C_k(S^1, S^1) \). Take any lifting \( \tilde{h} : \mathbb{R} \to \mathbb{R} \) of \( h \) and define the following homotopy

\[
H_{\tilde{h}} : \mathbb{R} \times [0, 1] \to \mathbb{R}, \quad H_{\tilde{h}}(x, t) = (1 - t)\tilde{h}(x) + t(\tilde{h}(0) + kx).
\]

Then it is easy to see that \( H_{\tilde{h}}(x + 1, t) = H_{\tilde{h}}(x, t) + k \) for all \( t, x \in \mathbb{R} \), and therefore it induces a certain homotopy \( H_{\tilde{h}} : S^1 \times [0, 1] \to S^1 \) between \( h \) and the map \( \phi_k(h(1)) : z \mapsto h(1) \cdot z^k \).

Notice that if \( \tilde{h}' = \tilde{h} + k \) is another lifting of \( h \), then the analogous homotopy \( H_{\tilde{h}'}(x, t) = (1 - t)(\tilde{h}'(x) + k) + t(\tilde{h}'(0) + kx) \) has the property that \( H_{\tilde{h}'} = H_{\tilde{h}} + k \), whence \( H_{\tilde{h}'} \) induces the same homotopy, i.e. \( H_{\tilde{h}'} = H_{\tilde{h}} \).

Hence \( H_{\tilde{h}} \) depends only on \( H \), and we will redenote it by \( H_h \).

One easily checks that the map \( H : C_k(S^1, S^1) \times [0, 1] \to C_k(S^1, S^1) \), \( H(h, t)(\cdot) = H_h(\cdot, t) \), is continuous with respect to the compact open topology on \( C_k(S^1, S^1) \). Moreover, if \( h \) belongs to the image of \( \phi_k \), then \( H_h \) is a constant homotopy at \( h \). Hence \( H \) is the desired strong deformation retraction of \( C_k(S^1, S^1) \) onto the image of \( \phi_k \).

Finally, if \( h \) is a homomorphism, then so is any of its liftings \( \tilde{h} \), and the corresponding homotopies \( H_{\tilde{h}} \) and \( H_h \) also consist of homeomorphisms. This implies that the inclusions \( \phi_{\pm 1} : O(2) \subset H(S^1) \) is a strong deformation retraction as well.
3) Statement (i) is a consequence of intermediate value theorem, and (ii) follows from (b).

(iii) \( \hat{h}_a \circ \hat{h}_b = \xi^a \circ \hat{h}_0 \circ \xi^b \circ \hat{h}_0 = \xi^a \circ \xi^{kb} \circ \hat{h}_0 = \xi^{a+kb} \circ \hat{h}_0. \)

(iv) If \( k = -1 \) then, by (iii), \( \hat{h}_a \circ \hat{h}_a = \xi^{a-a} \circ \hat{h}_0^2 = \hat{h}_0 \).

Hence one can put \( g := \hat{h}_0^2 = \hat{h}_a^2 \) and this map does not depend on \( a \in \mathbb{Z} \).

Let also \( \tilde{A} \) be the set of fixed points of \( g \). We have to show that \( \tilde{A} = p^{-1}(A) \).

Let \( s \in \tilde{A} \), and \( z = p(s) \). Then \( s = g^2(s) = \hat{h}_a^2(s) \) implies that

\[
z = p(s) = p \circ \hat{h}_a^2(s) = h^2 \circ p(s) = h^2(z),
\]

so \( z \in A \), that is \( p(\tilde{A}) \subset A \) and thus \( \tilde{A} \subset p^{-1}(A) \).

Conversely, let \( z \in A \) and \( s \in \mathbb{R} \) be such that \( z = p(s) \). Then there exists a unique lifting \( \hat{h}_a \) of \( h \) such that \( \hat{h}_a(s) = s \). But then \( g(s) = \hat{h}_a^2(s) = s \), whence \( s \in \tilde{A} \).

\( \square \)

The following example shows that the effect described in the statement (iv) of Lemma 5.1 includes the rigidity property of reflections of the circle mentioned in the introduction.

**Example 5.2.** Let \( h(z) = \bar{z}e^{-2\pi \phi}e^{2\pi \phi} = \bar{z}e^{2\phi} \) be a reflection of the complex plane with respect to the line passing though the origin and constituting an angle \( \phi \) with the positive direction of \( x \)-axis. Then \( h \) is an involution preserving the unit circle and the restriction \( h|_{S^1} : S^1 \to S^1 \) is a map of degree \(-1\) belonging to \( SO^{-}(2) \). Moreover, each its lifting \( \hat{h}_a : \mathbb{R} \to \mathbb{R} \) is given by \( \hat{h}_a(s) = a + \phi - s \). But then \( \hat{h}_a^2 = \text{id}_{\mathbb{R}^2} \) and does not depend on \( a \).

Moreover, the set of fixed points of \( \hat{h}_a^2 \) is \( \mathbb{R} \) which coincides with \( p^{-1}(S^1) \), where \( S^1 \) is the set of fixed points of \( h^2 = \text{id}_{S^1} \).

Another interpretation of the above results can be given in terms of shift functions.

**Corollary 5.3.** Let \( F : S^1 \times \mathbb{R} \to S^1 \) be a flow on the circle \( S^1 \) having no fixed points, so \( S^1 \) is a unique periodic orbit of \( F \) of some period \( \theta \).

1) Let \( h : S^1 \to S^1 \) be a continuous map. Then \( h \in C_1(S^1, S^1) \) if and only if there exists a continuous function \( \alpha : S^1 \to \mathbb{R} \) such that \( h = F^\alpha \). Such a function is not unique and is determined up to a constant summand \( n\theta \) for \( n \in \mathbb{Z} \). If \( F \) and \( h \) are \( C^r \), \( (0 \leq r \leq \infty) \), then so is \( \alpha \).

2) For every \( h \in C_{-1}(S^1, S^1) \) there exists a unique continuous function \( \alpha : S^1 \to \mathbb{R} \) such that

(a) \( h^2(z) = F(z, \alpha(z)) \) for all \( z \in S^1 \);

(b) \( \alpha(z) = 0 \) for some \( z \in S^1 \) iff \( h(z) = z \), (due to Lemma 5.1(i) there exists at least two such points);

(c) if \( F \) and \( h \) are \( C^r \), \( (0 \leq r \leq \infty) \), then so is \( \alpha \).
Let \( \alpha_k : S^1 \to \mathbb{R} \), \( k = 0,1 \), be two continuous functions, and for each \( t \in [0;1] \) let \( \alpha_t : S^1 \to \mathbb{R} \) and \( h_t : S^1 \to S^1 \) be defined by
\[
\alpha_t = (1 - t)\alpha_0 + t\alpha_1, \quad h_t(z) = ze^{2\pi i\alpha_t(z)}.
\]
If \( h_0 \) and \( h_1 \) are homeomorphisms (diffeomorphisms of class \( C^r \), \( 1 \leq r \leq \infty \)), then so is \( h_t \) for each \( t \in [0;1] \).

**Proof.** Let \( \tilde{F} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a unique lifting of \( F \) with respect to the covering \( p \). Then \( \mathbb{R} \) is a unique non-periodic orbit of \( \tilde{F} \).

(1) If \( \alpha : S^1 \to \mathbb{R} \) is any continuous function, then the map \( F_\alpha \) is homotopic to the identity by the homotopy \( \{ F_{t\alpha} \}_{t \in [0;1]} \), whence \( F_\alpha \) has degree 1 (the same as \( \text{id}_{S^1} \)).

Conversely, let \( h \in C_1(S^1,S^1) \) and let \( \tilde{h} \) be any lifting of \( h \). Then by Corollary 4.6 there exists a unique continuous function \( \beta : \mathbb{R} \to \mathbb{R} \) such that \( \tilde{h} = \tilde{F}_\beta \). Moreover, since \( h \) is a map of degree 1, we get from Lemma 5.1(b), that \( \tilde{h} \) commutes with \( \xi \), i.e. condition (g3) of Lemma 4.7 holds. Since \( \tilde{h} = \tilde{F}_\beta \) is a lifting of \( h \), Lemma 4.7(ii) implies that condition (g1) of Lemma 4.7 also holds, i.e. \( \beta \circ \xi = \beta \). Hence \( \beta \) induces a unique function \( \alpha : S^1 \to \mathbb{R} \) such that \( \beta = \alpha \circ p \) and \( \tilde{h} = \tilde{F}_{\alpha \circ p} \). Therefore By Lemma 4.7(2), \( \tilde{h} \) is a lifting of \( F_\alpha \). But it is also a lifting of \( h \), whence \( h = F_\alpha \).

(2) Since \( h \in C_{-1}(S^1,S^1) \), it follows that \( h^2 \in C_1(S^1,S^1) \), whence (a) and (c) directly follow from (1).

(b) Denote by \( A \) the set of fixed points of \( h^2 \). We should prove that \( A = \alpha^{-1}(0) \). Evidently, if \( \alpha(y) = 0 \), then
\[
h^2(y) = F(y,\alpha(y)) = h^2(y) = F(y,0) = y.
\]

Conversely, let \( y \in A \), so \( h(y) = y \), and let \( x \in \mathbb{R} \) be any point with \( p(x) = y \). Then there exists a unique lifting \( \tilde{h} \) of \( h \) with \( \tilde{h}(x) = x \) and by Lemma 5.1(iv), \( p^{-1}(A) \) is the set of fixed points of \( \tilde{h}^2 \). Moreover, since \( \tilde{h}^2 \) has degree 1, we get from (1) that \( \tilde{h}^2 = \tilde{F}_{\alpha \circ p} \). Since \( x \) is a fixed point of \( \tilde{h}^2 \) as well, we must have that \( 0 = \alpha \circ p(x) = \alpha(y) \).

(3) Consider two flows of \( \mathbb{R} \) and \( S^1 \) respectively:
\[
\tilde{F} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \tilde{F}(x,t) = x + t,
\]
\[
F : S^1 \times \mathbb{R} \to S^1, \quad F(z,t) = ze^{2\pi it}.
\]
Evidently, \( \tilde{F} \) is a lifting of \( F \), and \( h_t = F_{\alpha_t} \). Then by Lemma 4.7(2) the map
\[
\tilde{h}_t := \tilde{F}_{\alpha_t \circ p} : \mathbb{R} \to \mathbb{R}, \quad \tilde{h}_t(x) = x + (1 - t)\alpha_0(x) + t\alpha_1(x)
\]
is a lifting of \( h_t \). Evidently, \( \tilde{h}_t = (1 - t)\tilde{h}_0 + t\tilde{h}_1 \). Since \( \tilde{h}_0 \) and \( \tilde{h}_1 \) are homeomorphisms (diffeomorphisms of class \( C^r \)), it follows that so are \( h_0 \) and
\( \tilde{h}_1 \) are homeomorphisms. Therefore so are their convex linear combination \( \tilde{h}_t \) and the induced map \( h_t \). \( \square \)

6. Flows without fixed points

Let \( X \) be a Hausdorff topological space and \( F : Y \times \mathbb{R} \to Y \) a continuous flow on an open subset \( Y \subset X \times S^1 \) satisfying the following conditions:

(\( \Phi 1 \)) the orbits of \( F \) are exactly the connected components of the intersections \( (x \times S^1) \cap Y \) for all \( x \in X \).

(\( \Phi 2 \)) \( F \) admits flow box charts at each point \( (y, s) \in Y \);

(\( \Phi 3 \)) the set \( B = \{ x \in X \mid x \times S^1 \subset Y \} \) is dense in \( X \);

(\( \Phi 4 \)) for each \( x \in X \), the intersection \( (x \times S^1) \cap Y \) has only finitely many connected components (being by (\( \Phi 1 \)) orbits of \( F \)).

Thus every orbit of \( F \) is either \( x \times S^1 \) or some arc in \( x \times S^1 \), and, in particular, \( F \) has no fixed points. Also condition (\( \Phi 3 \)) implies that the set of periodic orbits of \( F \) is dense in \( X \times S^1 \). Then the complement to \( B \):

\[ A := X \setminus B = \{ x \in X \mid x \times S^1 \not\subset Y \} \]

consists of \( x \in X \) for which \( x \times S^1 \) contains a non-closed orbit of \( F \).

It will also be convenient to use the following notations for \( x \in X \):

\[ L_x := (x \times S^1) \cap Y, \quad \tilde{L}_x := p^{-1}(L_x) = (x \times \mathbb{R}) \cap \tilde{Y}. \]

**Example 6.1.** Let \( X \) be a smooth manifold, and \( G \) be a vector field on \( X \times S^1 \) defined by \( G(x, s) = \frac{\partial}{\partial s} \), so its orbits are the circles \( x \times S^1 \). Let \( Y \subset X \times S^1 \). Then it is well known that there exists a non-negative \( C^\infty \) function \( \alpha : X \times S^1 \to [0; +\infty) \) such that \( Y = (X \times S^1) \setminus \alpha^{-1}(0) \). Define another vector field \( F \) on \( X \times S^1 \) by \( F = \alpha G \). Let also

\[ F : (X \times S^1) \times \mathbb{R} \to X \times S^1 \]

be the flow on \( X \times S^1 \) generated by \( F \). Then \( Y \) is the set of non-fixed points of \( F \) and the induced flow on \( Y \) satisfies conditions (\( \Phi 1 \)) and (\( \Phi 2 \)). Another two conditions (\( \Phi 3 \)) and (\( \Phi 4 \)) depend only on a choice of \( Y \). For instance they will hold if the complement \((X \times S^1) \setminus Y \) is finite.

**Theorem 6.2.** Suppose \( F \) satisfies (\( \Phi 1 \))- (\( \Phi 4 \)). Let also \( h : Y \to Y \) be a homeomorphism having the following properties.

1. \( h(L_x) \subset x \times S^1 \) for each \( x \in X \), so \( h \) preserves the first coordinate, and in particular, leaves invariant each periodic orbit of \( F \), though it may interchange non-periodic orbits contained in \( x \times S^1 \).

2. For each \( x \in B \) the restriction \( h : x \times S^1 \to x \times S^1 \) has degree \(-1\) as a self-map of a circle.

Then there exists a unique continuous function \( \alpha : Y \to \mathbb{R} \) such that
(a) \( h^2(x, s) = F(x, s, \alpha(x, s)) \) for all \((x, s) \in Y\), in particular \( h^2 \) preserves every orbit of \( F \);

(b) \( \alpha(x, s) = 0 \) iff \( h(x, s) = (x, s) \);

(c) if in addition \( X \) is a manifold of class \( C^r \), \((0 \leq r \leq \infty)\), \( F \) is \( C^r \) and admits \( C^r \) flow box charts, and \( h \) is \( C^r \), then \( \alpha \) is \( C^r \) as well.

**Proof.** Let \( p : X \times \mathbb{R} \to X \times S^1 \) be the infinite cyclic covering of \( X \times S^1 \) defined by \( p(x, s) = (x, e^{2\pi is}) \) and \( \xi(x, s) = (x, s + 1) \) be a diffeomorphism of \( X \times \mathbb{R} \) generating the group \( \mathbb{Z} \) of covering slices. In particular, \( p \circ \xi = p \).

Denote \( \tilde{Y} = p^{-1}(Y) \) Then the restriction \( p : \tilde{Y} \to Y \) is a covering of \( Y \).

Let \( \tilde{F} : \tilde{Y} \times \mathbb{R} \to \tilde{Y} \) be the lifting of the flow \( F \), so

\[
p \circ \tilde{F}(y, t) = F(p(y), t), \quad (y, t) \in \tilde{Y} \times \mathbb{R}.
\]

Then the orbits of \( \tilde{F} \) are exactly the connected components of \( \tilde{L}_x \) for all \( x \in X \). In particular, all orbits of \( \tilde{F} \) are non-closed. Evidently, for \( x \in X \) the following conditions are equivalent:

- \( x \times \mathbb{R} \) is an orbit of \( \tilde{F} \);
- \( x \times \mathbb{R} \subseteq \tilde{Y} \);
- \( x \times S^1 \) is an orbit of \( F \);
- \( x \times S^1 \subseteq Y \), i.e. \( x \in B \).

**Lemma 6.2.1.** Let \( \tilde{h} : \tilde{Y} \to \tilde{Y} \) be any lifting of \( h \), that is \( p \circ \tilde{h} = h \circ p \). Then the following conditions hold.

(i) \( \tilde{h}(\tilde{L}_x) \subseteq \tilde{L}_x \) for all \( x \in X \);

(ii) if \( \tilde{h}_1 \) is another lifting of \( h \), then \( \tilde{h}_1^2 = \tilde{h}^2 \);

(iii) for each \( x \in X \) the restriction \( \tilde{h} : \tilde{L}_x \to \tilde{L}_x \) is strictly decreasing in the sense that if \( h(x, s) = (x, t) \) for some \( s, t \in \mathbb{R} \), then \( s > t \);

(iv) for each \( x \in X \) the restriction \( \tilde{h}^2 \) leaves invariant each orbit of \( \tilde{F} \).

**Proof.** (i) Notice that \( p \circ \tilde{h}(\tilde{L}_x) = h \circ p(\tilde{L}_x) = h(L_x) \subseteq L_x \), whence

\[
\tilde{h}(\tilde{L}_x) \subseteq p^{-1}(h(L_x)) \subseteq p^{-1}(L_x) = \tilde{L}_x.
\]

(ii) Let \( x \in B \), so \( x \times S^1 \subseteq Y \) is an orbit of \( F \). Then the restriction \( p|_{x \times \mathbb{R}} : x \times \mathbb{R} \to x \times S^1 \) is a universal covering map, \( h|_{x \times S^1} : x \times S^1 \to x \times S^1 \) is a map of degree \(-1\), and \( \tilde{h}|_{x \times \mathbb{R}}, \tilde{h}_1|_{x \times \mathbb{R}} : x \times \mathbb{R} \to x \times \mathbb{R} \) are two lifting of \( h|_{x \times S^1} \). Hence by Lemma 5.1(iv) \( \tilde{h}^2|_{x \times \mathbb{R}} = \tilde{h}_1^2|_{x \times \mathbb{R}} \). Thus \( \tilde{h}^2 = \tilde{h}_1^2 \) on \( B \times \mathbb{R} \). But by property (Φ3), \( B \) is dense in \( X \), whence \( B \times \mathbb{R} \) is dense in \( X \times \mathbb{R} \). Since \( X \) is Hausdorff, it follows that \( \tilde{h}^2 = \tilde{h}_1^2 \) on all of \( X \times \mathbb{R} \).

(iii) If \( x \in B \), then \( h|_{x \times S^1} : x \times S^1 \to x \times S^1 \) is a map of degree \(-1\), whence \( \tilde{h}|_{x \times \mathbb{R}} : x \times \mathbb{R} \to x \times \mathbb{R} \) reverses orientation of \( x \times \mathbb{R} \), i.e. is strictly decreasing.
Suppose $x \in X \setminus B$, i.e. $x \times \mathbb{R} \not\subset \tilde{Y}$. We need to show that $\tilde{h}|_{\tilde{L}_x} : \tilde{L}_x \to \tilde{L}_x$ is strictly decreasing, that is if $(x, s_0), (x, s_1) \in \tilde{L}_x$ are two distinct points with $s_0 < s_1$, $(x, t_0) = \tilde{h}(x, s_0)$, and $(x, t_1) = \tilde{h}(x, s_1)$, then $t_0 > t_1$.

Since $\tilde{h}$ is a homeomorphism, it follows that $t_0 \neq t_1$. Suppose that $t_0 < t_1$. Then there exist $a > 0$, and two open neighborhoods $V \subset U$ of $x$ in $X$ such that

$$\tilde{h}(V \times s_i) \subset U \times (t_i - a; t_i + a), \quad i = 0, 1,$$

$$t_0 + a < t_1 - a. \quad (6.1)$$

Due to property $(\Phi 3)$, the set $B$ is dense in $X$, so there exists a point $y \in B \cap V \neq \emptyset$.

Then on the one hand $y \times \mathbb{R} \subset \tilde{Y}$ is an orbit of $\tilde{F}$, and $\tilde{h} : y \times \mathbb{R} \to y \times \mathbb{R}$ reverses orientation, so if $(y, t'_i) = \tilde{h}(y, s_i)$, $i = 0, 1$, then $t'_0 > t'_1$.

On the other hand, due to (6.1) $t'_i \in (t_i - a; t_i + a)$, whence by (6.2):

$$t'_0 < t_0 + a < t_1 - a < t'_1$$

which gives a contradiction. Hence $t_0 > t_1$.

(iv) If $x \in B$, then $\tilde{L}_x = x \times \mathbb{R}$ is an orbit of $\tilde{F}$ and by (i) it is invariant with respect to $\tilde{h}$. Hence it is also invariant with respect to $\tilde{h}^2$.

Suppose $x \in X \setminus B$. Then by property $(\Phi 4)$, if $L_x = (x \times S^1) \cap Y$ consists of finitely many connected components $I_0, \ldots, I_{n-1}$ for some $n$ enumerated in the cyclical order along the circle $x \times S^1$. This implies that $\tilde{L}_x$ is a disjoint union of countably many open intervals $I_i$, $(i \in \mathbb{Z})$, being orbits of $\tilde{F}$ which can be enumerated so that $\xi(\tilde{I}_k) = \tilde{I}_{k+n}$ and $p(\tilde{I}_k) = I_{k \mod n}$.

Since $\tilde{h}$ is a strictly decreasing homeomorphism of $\tilde{L}_x$, it follows that there exists $a \in \mathbb{Z}$ such that $\tilde{h}^{a}(\tilde{I}_k) = \tilde{I}_{a-k}$. Hence

$$\tilde{h}^2(\tilde{I}_k) = \tilde{h}(\tilde{I}_{a-k}) = \tilde{I}_{a-(a-k)} = \tilde{I}_k.$$  \hfill \square

Now we can deduce our Theorem from Lemma 6.2.1.

(a) By Corollary 4.8 applied to $h^2$ there exists a unique continuous function $\alpha : Y \to \mathbb{R}$ such that

$$h^2(x, s) = \tilde{F}(x, s, \alpha \circ p(x, s)), \quad h^2(x, s) = F(x, s, \alpha(x, s)).$$

(b) Let $(x, s) \in X \times S^1$. If $x \in B$, then $x \times S^1 \subset Y$ is an orbit of $\tilde{F}$ and by Corollary 5.3(b) $\alpha(x, s) = 0$ iff $h^2(x, s) = (x, s)$.

Suppose $x \in X \setminus B$. Then $(x, s)$ is a non-periodic point of $\tilde{F}$, whence $h^2(x, s) = F(x, s, \alpha(x, s)) = (x, s)$ is possible if and only if $\alpha(x, s) = 0$.

(c) Smoothness properties of $\alpha$ follows similarly to the statement (c) of Corollary 4.8. \hfill \square
7. Polar coordinates

Let $\mathbb{H} = \mathbb{R} \times [0; +\infty)$, $\text{Int} (\mathbb{H}) = \mathbb{R} \times (0; +\infty)$, and

$$p : \mathbb{H} \to \mathbb{C} \equiv \mathbb{R}^2, \quad p(\rho, \phi) = \rho e^{2\pi i \phi} = (\rho \cos \phi, \rho \sin \phi),$$

be the infinite branched covering map defining polar coordinates.

**Lemma 7.1.** ([9, Lemma 11.1]) Let $U \subset \mathbb{C}$ be an open neighborhood of 0, and $h : U \to \mathbb{C}$ a $C^r$, $1 \leq r \leq \infty$, smooth embedding with $h(0) = 0$. Then there exists a $C^{r-1}$ embedding $\tilde{h} : p^{-1}(U) \to \mathbb{H}$ such that $p \circ \tilde{h} = h \circ p$.

Suppose, in addition, that the Jacobi matrix $J(h, 0)$ of $h$ at 0 is orthogonal. Then there exists $a \in \mathbb{R}$ such that for each $s \in \mathbb{R}$

$$\tilde{h}(0, s) = \begin{cases} (0, a + s) & \text{if } J(h, 0) \in \text{SO}(2), \\ (0, a - s) & \text{if } J(h, 0) \in \text{SO}^{-}(2). \end{cases}$$

Hence in the second case (when $h$ reverses orientation) $\tilde{h}^2(0, s) = (0, s)$, that is $\tilde{h}$ is always fixed on $0 \times \mathbb{R}$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial without multiple factors and having degree $k \geq 2$ and $F = -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$ be the Hamiltonian vector field of $f$. Since the restriction $p : \text{Int} (\mathbb{H}) \to \mathbb{R}^2 \setminus 0$ is a infinite cyclic covering map, $F$ induces a vector field $\tilde{F}$ on $\text{Int} (\mathbb{H})$. One can even obtain precise formulas for $\tilde{F}$ (see [6, §4.2, Corollary 4.4]):

$$\tilde{F}(r, \phi) = \left( \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi} \right) \begin{pmatrix} \cos \phi & \sin \phi \\ -\frac{1}{\rho} \sin \phi & \frac{1}{\rho} \cos \phi \end{pmatrix} \begin{pmatrix} -f_y'(z) \\ f_x'(z) \end{pmatrix},$$

where $z = p(\rho, \phi) = \rho e^{2\pi i \phi}$. The latter formula can be reduced to a more simplified form. Define the following function $q : \mathbb{C}\setminus 0 \to \mathbb{C}$ by

$$q(z) = \frac{-f_y'(z) + if_x'(z)}{z} = \frac{(-f_y'(z) + if_x'(z))\overline{z}}{|z|^2}.$$

Then

$$\tilde{F}(r, \phi) = \text{Re}(q(z)) \frac{\partial}{\partial \rho} + \text{Im}(q(z)) \frac{1}{\rho} \frac{\partial}{\partial \phi}.$$

Since $f$ is a homogeneous of degree $k \geq 2$, it follows that $\tilde{F}(r, \phi)$ smoothly extends to $\mathbb{H}$.

Let $F$ and $\tilde{F}$ be the local flows generated by $F$ and $\tilde{F}$ respectively. Then $F_t \circ p(\rho, \phi) = p \circ \tilde{F}_t(\rho, \phi)$ whenever all parts of that identity are defined.

**Example 7.2.** 1) Suppose 0 is a non-degenerate local extreme of $f$. Then one can assume that $f(x, y) = \frac{1}{2}(x^2 + y^2)$. In this case

$$F(x, y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad F(z, t) = z e^{2\pi i t},$$
\[ \tilde{F}(\rho, \phi) = \frac{\partial}{\partial \phi}, \quad \tilde{F}(\rho, \phi, t) = (\rho, \phi + t). \]

2) Let 0 be a non-degenerate saddle, so one can assume that \( f(x, y) = xy \).

Then
\[
F(x, y) = -x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad F(x, y, t) = (xe^{-t}, ye^t),
\]
\[ \tilde{F}(\rho, \phi) = \rho \cos 2\phi \frac{\partial}{\partial \rho} + \sin 2\phi \frac{\partial}{\partial \phi}. \]

Notice that writing down precise formulas for \( \tilde{F} \) is a rather complicated task.

3) If 0 is a degenerate critical point of \( f \), so \( \deg f \geq 3 \), then the situation is more complicated. Notice that in this case \( \tilde{F} \) is zero on \( \partial \mathbb{H} = \mathbb{R} \times 0 \), whence \( \tilde{F} \) is fixed on that line. Again the formulas for \( F \) and \( \tilde{F} \) are highly complicated.

**Lemma 7.3.** (see [6, Theorem 1.6], [9, Proof of Theorem 5.6]) Let \( U \subset \mathbb{R}^2 \) be an open neighborhood of the origin \( 0 \in \mathbb{R}^2 \), \( h : U \to \mathbb{R}^2 \) an embedding which preserves orbits of \( F \), and \( \alpha : U \setminus 0 \to \mathbb{R} \) be a \( C^\infty \) function such that \( h(x) = F(x, \alpha(x)) \) for all \( x \in U \setminus 0 \). Let also \( \tilde{h} : p^{-1}(U) \to \mathbb{H} \) be any lifting of \( h \) as in Lemma 7.1. Then \( \alpha \) can be defined at \( x \) so that it becomes \( C^\infty \) in \( U \) in the following cases:

(a) \( x \) is a non-degenerate local extreme of \( f \) or a (possibly degenerate) saddle point;
(b) \( x \) is a degenerate local extreme of \( f \) and \( \tilde{h} \) is fixed on \( \mathbb{R} \times 0 \).
(c) \( x \) is a degenerate local extreme of \( f \) and there exists an open neighborhood \( V \subset U \) and another embedding \( q : U \to \mathbb{R}^2 \) such that \( q(V) \subset U \), \( q \) preserves orbits of \( F \) and reverses their orientation, and \( h = q^2 \).

**Proof.** Cases (a) and (b) are proved in [6, Theorem 1.6], see also proof of Theorem 5.6 in [9].

(c)⇒(b) Let \( \tilde{q} : p^{-1}(U) \to \mathbb{H} \) be any lifting of \( q \) as in Lemma 7.1. Then \( \tilde{q}^2 \) is a lifting of \( q^2 = h \). Moreover, since \( \tilde{q} \) reverse orientation of orbits, we get from Lemma 7.1 that \( \tilde{q}^2 \) is fixed on \( \mathbb{R} \times 0 \). Hence condition (b) holds. \( \square \)

8. **CHIPPED CYLINDERS OF A MAP \( f \in \mathcal{F}(M, P) \)**

Let \( f \in \mathcal{F}(M, P) \). In what follows we will use the following notations.

(i) \( K_1, \ldots, K_k \) denote all the critical leaves of \( f \), and
\[ \mathcal{K} = \bigcup_{i=1}^{k} K_i. \]

(ii) Let \( R_{K_i}, i = 1, \ldots, k \), be an \( f \) regular neighborhood of \( K_i \) chosen so that \( R_{K_i} \cap R_{K_j} = \emptyset \) for \( i \neq j \).
(iii) Let also $L_1, \ldots, L_l$ be all the connected components of $M \setminus K$;

(iv) For each $i = 1, \ldots, l$ let

$$
\mathcal{N}_i = L_i \setminus \Sigma_f.
$$

Then there exist a finite subset $Q_i \subset \{-1, 1\} \times S^1$, and an immersion $\phi_i : ([{-1,1} \times S^1) \setminus Q_i \to \mathcal{N}_i$ and a $C^\infty$ embedding $\eta : [0,1] \to P$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
([0; 1] \times S^1) \setminus Q_i & \xrightarrow{\phi} & \mathcal{N}_i \\
\downarrow p_1 & & \downarrow f \\
[0; 1] & \xrightarrow{\eta} & P
\end{array}
$$

where $p_1$ is the projection to the first coordinate. Notice that $\phi$ can be non-injective only at points of $\{-1,1\} \times S^1$ and this can happens only when $P = S^1$, see Example 8.1 and Figure 8.2d) below.

We will call $\mathcal{N}_i$ a chipped cylinder of $f$, see Figure 8.1.

![Figure 8.1](image)

It will also be convenient to denote

$$
\mathcal{N}_i^- = \phi_i([-1;0] \times S^1) \setminus Q_i), \quad \mathcal{N}_i^+ = \phi_i([0; 1] \times S^1) \setminus Q_i),
$$

$$
\text{Int}\mathcal{N}_i = \phi_i((-1; 1) \times S^1).
$$

We will call $\mathcal{N}_i^-$ and $\mathcal{N}_i^+$ chipped half-cylinders of $\mathcal{N}_i$ and $f$, and $\text{Int}\mathcal{N}_i$ the interior of $\mathcal{N}_i$.

(v) Let also

$$
Z_i = \mathcal{N}_i \cup \left(\bigcup_{\mathcal{N}_i \cap K_j \neq \emptyset} R_{K_j}\right)
$$

be the union of the chipped cylinder $\mathcal{N}_i$ with $f$-regular neighborhoods of critical leaves of $f$ which intersect the closure $\overline{\mathcal{N}_i}$. We will call $Z_i$ an $f$-regular neighborhood of $\mathcal{N}_i$. 
Example 8.1. a) Let \( f : [0,1] \times S^1 \to \mathbb{R} \) be given by \( f(s,x) = s \), so it has no critical points, see Figure 8.2a). Then it has a unique chipped cylinder \( N = [0,1] \times S^1 \), which coincides with its \( f \)-regular neighborhood and \( K = \emptyset \).

b) Let \( f : D^2 \to \mathbb{R} \) be a function of class \( \mathcal{F}(D^2, \mathbb{R}) \) having only one critical point \( z \), see Figure 8.2b). Then \( K = \{z\} \), \( f \) has a unique chipped cylinder \( N = D^2 \setminus \{z\} \), and its \( f \)-regular neighborhood is all \( D^2 \).

c) Let \( f : S^2 \to \mathbb{R} \) be a function of class \( \mathcal{F}(S^2, \mathbb{R}) \) having only two critical points \( z_1 \) and \( z_2 \) being therefore extremes of \( f \), see Figure 8.2c). Then \( K = \{z_1, z_2\} \), \( f \) has a unique chipped cylinder \( N = S^2 \setminus \{z_1, z_2\} \) and its \( f \)-regular neighborhood is all \( S^2 \).

d) Let \( M \) be either a 2-torus or Klein bottle with a hole and \( f : M \to S^1 \) be a map of class \( \mathcal{F}(M,P) \) schematically shown in Figure 8.2d). It has only one critical point \( z \) and that point is a saddle, a unique critical leaf \( K = K \), and two chipped cylinders \( N_1 \) and \( N_2 \). It follows from the Figure 8.2d) that \( \overline{N}_1 \) intersects only one \( K \) from “both sides”, in the sense that both intersections \( \overline{N}_1^- \cap K \) and \( \overline{N}_1^+ \cap K \) are non-empty.

e) Let \( f : [0,1] \times S^1 \to \mathbb{R} \) be a Morse function having one minimum \( z \) and one saddle point \( y \) as in Figure 8.3. Then \( f \) has two critical leaves: the point \( z \) and a critical leaf \( K \) containing \( y \), and three chipped cylinders \( N_1, N_2, N_3 \). Let \( R_K \) be an \( f \)-regular neighborhood of \( K \). Then the corresponding \( f \)-regular neighborhoods of chipped cylinders are the following ones:

\[
Z_1 = N_1 \cup \{z\} \cup R_K, \quad Z_2 = N_2 \cup R_K, \quad Z_3 = N_3 \cup R_K.
\]

The following lemma describes simple properties of chipped cylinders. The proof is left for the reader.
Lemma 8.2. Let $N$ be a chipped cylinder of $f$, $N^-$ and $N^+$ be its chipped half-cylinders, and $p : M \to \Gamma_f$ be the projection onto the graph of $f$. Then the following statements hold.

1. $N^-$ and $N^+$ are orientable manifolds. Moreover, if $P = \mathbb{R}$, then $N$ is orientable as well. However, if $P = S^1$, then it is possible to construct an example of $f$ having non-orientable chipped cylinder, see Example 8.1d).

2. Each of the closures $\overline{N^-}$ and $\overline{N^+}$ intersects at most one critical leaves of $f$, and those intersections consist of open arcs being leaves of the singular foliation $\Xi_f$.

3. If $\overline{N} \cap K_i = \emptyset$ but $\overline{N} \cap K_i = \emptyset$, then $K_i$ is a critical point of $f$ being a local extreme.

4. $N' \cap N'' \subset \mathcal{K}$ for any two distinct chipped cylinders $N'$ and $N''$ of $f$.

5. Every regular leaf of $f$ is contained in some chipped cylinder of $f$.

The following theorem is the principal technical tool. Let $M$ be a (possibly non-orientable) compact surface, $f \in \mathcal{F}(M, P)$, $N$ a chipped cylinder of $f$, and $Z = N \cup \bigcup_{K_j \neq \emptyset} R_{K_j}$ be its $f$-regular neighborhood of $N$.

Theorem 8.3. 1) Let $V \subset N$ be a regular leaf of $f$, and $h \in S(f|_Z)$ a diffeomorphism of $Z$ such that $h(V) = V$ and $h$ reverses orientation of $V$. Then every leaf of the singular foliation $\Xi_f$ in $Z$ is $(h^2, +)$-invariant, i.e. $h^2$ fixes all critical points of $f$ in $Z$ and preserves all other leaves of $\Xi_f$ in $Z$ with their orientations.

2) Suppose in addition that $Z$ is orientable and let $F$ be any Hamiltonian like flow of $f|_Z$ on $Z$. Then there exists a unique $C^\infty$ function $\alpha : Z \to \mathbb{R}$ such that $h^2 = F_\alpha$ on $Z$ and $\alpha = 0$ at each fixed point of $h^2$ in $\text{Int} N$ and for each local extreme $z$ of $f$ in $Z$.

Proof. 1) Let us mention that since $h$ reverses orientation of $V$, it reverses orientations of all regular leaves in $N$. Therefore those leaves are $(h^2, +)$-invariant, and we should prove the same for all other leaves of $\Xi_f$ in $Z$.

First we introduce the following notation. If $\mathcal{K} \cap \overline{N^-} = \emptyset$, then let $K^-$ be a unique critical leaf of $f$ intersecting $\overline{N^-}$, and let $R_{K^-}$ be its $f$-regular neighborhood. Otherwise, when $\mathcal{K} \cap \overline{N^-} = \emptyset$, put $K^- = R_{K^-} = \emptyset$, Define further $K^+$ and $R_{K^+}$ in a similar way with respect to $\overline{N^+}$. Then

$$Z = R_{K^-} \cup N^- \cup N^+ \cup R_{K^+}.$$  

As those four sets are invariant with respect to $h$, it suffices to prove that $h^2$ preserves leaves of $\Xi_f$ with their orientation for each of those sets.
1a) Let us show that $h^2$ preserves all leaves of $\Xi_f$ in $\mathcal{N}^-$.

Since $\mathcal{N}^-$ is an orientable manifold, one can construct a Hamiltonian like flow of $f$ on $\mathcal{N}^-$. Evidently, $F$ satisfies conditions $(\Phi_1)$-$(\Phi_4)$. Moreover, since $h$ reverses orientation of all periodic orbits of $F$ in $\mathcal{N}^+$, we get from Theorem 6.2 that there exists a unique $C^\infty$ function $\alpha : \mathcal{N} \to \mathbb{R}$ such that $h^2|_{\mathcal{N}^-} = F_\alpha$ and $\alpha$ vanishes at fixed points of $h^2$ on regular leaves of $f$ in $\mathcal{N}^-$. In particular, each non-periodic orbit of $F$ in $\mathcal{N}^+$ is $(h^2, +)$-invariant as well.

1b) Now let us prove that each leaf of $\Xi_f$ in $K^-$ is also $(h^2, +)$-invariant. This will imply $(h^2, +)$-invariantness of all leaves of $\Xi_f$ in $R_{K^-}$ (see the proof of the implication (iii)$\Rightarrow$(i) in [9, Lemma 7.4]).

If $K^-$ is empty there is nothing to prove.

If $K^- \cap \mathcal{N}^- = \emptyset$, then by Lemma 8.2(3) $K^-$ is a local extreme of $f$. Hence $K$ is an element of $\Xi_f$ and its is evidently invariant with respect to $h$, and therefore with respect to $h^2$.

Thus assume that $K^- \cap \mathcal{N}^- \neq \emptyset$. Then this intersection contains a non-periodic orbit $\gamma$ of $F$. Since $\gamma$ is $(h^2, +)$-invariant, it follows from [7, Claim 7.1.1] or [9, Lemma 7.4], that all elements of the foliation $\Xi_f$ are also $(h^2, +)$-invariant.

Let us recall a simple proof of that fact. Indeed, let $v$ be a vertex of $\gamma$ being therefore a critical point of $f$. Then $h(v) = v$, whence $h^2$ preserves the set of all edges incident to $v$. Moreover, as $h^2$ preserves orientation at $v$, it must also preserve cyclic order of edges incoming to $v$. But since $\gamma$ (being one of those edges) is $(h^2, +)$-invariant, it follows that so are all other edges incident to $v$. Applying the same arguments to those edges and so on, we will see that $h^2$ preserves all edges of $K$ with their orientation.

The proofs for $\mathcal{N}^+$ and $R_{K^+}$ are similar.

2) Assume now that $Z$ is an orientable surface and let $F$ be any Hamiltonian like flow of $f$. We know from 1) that $h^2$ preserves all orbits of $F$ with their orientations.

2a) We claim that there exists a unique $C^\infty$ function $\alpha : \mathcal{N} \to \mathbb{R}$ such that $h^2|_{\mathcal{N}} = F_\alpha$ and $\alpha$ vanishes at fixed points of $h^2$ on periodic orbits.

If both $K^-$ and $K^+$ are non-empty and distinct, then the restriction of $F$ to $\mathcal{N}$ satisfies conditions $(\Phi_1)$-$(\Phi_4)$. Moreover, as $h$ reverses orientation of all periodic orbits of $F$, the statement follows from Theorem 6.2 as in 1a).

However, if $K^- = K^+$, as in Example 8.1d), the situation is slightly more complicated: $\mathcal{N}$ might be not of the form $([-1, 1] \times S^1) \setminus Q$ for some finite set $Q \subset \{-1, 1\} \times S^1$ and Theorem 6.2 is not directly applicable. Nevertheless, one can apply that theorem to each of the sets $\mathcal{N}^-, \mathcal{N}^+$, and
Int $\mathcal{N}$ and construct three functions

$$\alpha^- : \mathcal{N}^- \to \mathbb{R}, \quad \alpha^- : \mathcal{N}^+ \to \mathbb{R}, \quad \alpha^0 : \text{Int} \mathcal{N} \to \mathbb{R}$$

satisfying $h^2|_{\mathcal{N}^-} = F_{\alpha^-}$, $h^2|_{\mathcal{N}^+} = F_{\alpha^+}$, $h^2|_{\text{Int} \mathcal{N}} = F_{\alpha^0}$, and vanishing at fixed points of $h^2$ on periodic orbits. From uniqueness of such functions, we get that $\alpha^- = \alpha^0$ on $\mathcal{N}^- \cap \text{Int} \mathcal{N}$ and $\alpha^+ = \alpha^0$ on $\mathcal{N}^+ \cap \text{Int} \mathcal{N}$.

A possible problem is that $\mathcal{N}$ intersects $K^-$ from “both sides”, and therefore a priori $\alpha^+$ and $\alpha^-$ can differ on $\mathcal{N}^- \cap \mathcal{N}^+ \cap K^-$. However, $\mathcal{N}^- \cap \mathcal{N}^+ \cap K^-$ consists of non-periodic orbits of $F$, and therefore for each such orbit $\gamma$ the identity $h^2|_\gamma = F_{\alpha^-}|_\gamma = F_{\alpha^+}|_\gamma$ implies that $\alpha^- = \alpha^+$ on $\gamma$.

Thus $\alpha^- = \alpha^+$ on $\mathcal{N}^- \cap \mathcal{N}^+ \cap K^-$, and therefore those functions define a well defined $C^\infty$ function $\alpha : \mathcal{N} \to \mathbb{R}$ satisfying $h^2|_{\text{Int} \mathcal{N}} = F_\alpha$ and $\alpha$ vanishes at fixed points of $h^2$ on periodic orbits.

2b) It remains to show that $\alpha$ extends to a shift function for $h^2$ on $R_{K^-} \cup R_{K^+}$ and thus on all of $Z$. It suffices to prove that for $R_{K^-}$.

If $K^- = \emptyset$, then $R_{K^-} = \emptyset$ and there is nothing to prove.

If $K^-$ is a local extreme of $f$, then by Lemma 7.3 (cases (a) and (c) for non-degenerate and degenerate critical point) $\alpha$ can be defined at $K^-$ so that it becomes $C^\infty$.

In all other cases $K^-$ contains a non-periodic orbit of $F$. Then by the implication (ii)$\Rightarrow$(iv) of [9, Lemma 7.4], $\alpha$ extends to a $C^\infty$ shift function for $h^2$ on $R_{K^-}$. It remains to prove the following lemma:

**Lemma 8.3.1.** $\alpha(z) = 0$ for every local extreme of $f$ in $Z$.

**Proof.** Indeed, it is evident, that arbitrary small neighborhood of $z$ contains a periodic orbit $\gamma$ of $F$. Since $h$ reverses orientations of $\gamma$, we have from Lemma 5.1(i) that $h$ always has at least one fixed point $x \in \gamma$ (in fact it has even two such points). Hence by Corollary 5.3(b), $\alpha(x) = 0$. Then by continuity of $\alpha$ we should have that $\alpha(z) = 0$ as well. □

Theorem 8.3 is completed. □

9. CREATING ALMOST PERIODIC DIFFEOMORPHISMS

Let $M$ be a compact orientable surface, $f \in \mathcal{F}(M, P)$, $Z$ be an $f$-adapted subsurface, $h \in \mathcal{S}(f)$ be such that $h(Z) = Z$, and $m \geq 2$. If $h^m|_Z$ is isotopic to the identity of $Z$ by $f$-preserving isotopy, then the following Lemma 9.1 gives conditions when one can change $h$ on $M \backslash Z$ so that its $m$-power will be $f$-preserving isotopic to the identity on all of $M$. The proof follows the line of [9, Lemma 13.1(3)] in which $M$ is a 2-disk or a cylinder.
Lemma 9.1. (cf. [9, Lemma 13.1(3)]) Let $M$ be a compact orientable surface, $f \in \mathcal{F}(M, P)$, $Z$ be an $f$-adapted subsurface, and $h \in \mathcal{S}(f)$ be such that $h(Z) = Z$. Suppose that the following conditions hold.

1. Each component of $Z$ contains at least one saddle point of $f$.
2. There exists $m \geq 2$ and $a \geq 1$ such that connected components of $M \setminus Z$ can be enumerated as follows:

\[
\begin{array}{cccc}
Y_{1,0} & Y_{1,1} & \cdots & Y_{1,m-1} \\
Y_{2,0} & Y_{2,1} & \cdots & Y_{2,m-1} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{a,0} & Y_{a,1} & \cdots & Y_{a,m-1}
\end{array}
\] (9.1)

so that $h(Y_{i,j}) = Y_{i,j+1 \mod m}$ for all $i, j$, that is $h$ cyclically shifts columns in (9.1).

3. $h^m$ is isotopic in $\mathcal{S}(f)$ to a diffeomorphism $\tau$ fixed on some neighborhood of $Z$ (by [9, Lemma 7.1]) this condition holds if there exists $C^\infty$ function $\alpha : Z \to \mathbb{R}$ such that $h^m|_Z = \mathbf{F}_\alpha$.

Then there exists $g \in \mathcal{S}(f)$ such that $g = h$ on $Z$ and $g^m \in \mathcal{S}_{id}(f)$, that is $g^m = \mathbf{F}_\beta$ for some $C^\infty$ function $\beta : M \to \mathbb{R}$.

Proof. Let $Y_j = \bigcup_{i=1}^a Y_{i,j}$, $j = 0, \ldots, m-1$, be the union of components from the same column of (9.1). Then $h(Y_j) = Y_{j+1 \mod m}$. Notice that condition (2) implies that $h^m(Y_{i,j}) = Y_{i,j}$ for all $i, j$.

We will show that the desired diffeomorphism $g \in \mathcal{S}(f)$ can be defined by the formula:

\[
g(x) = \begin{cases} h(x), & x \in Z \cup Y_0 \cup \cdots \cup Y_{m-2}, \\ \tau^{-1} \circ h(x), & x \in Y_{m-1}. \end{cases}
\]

Indeed, by definition $g = h$ on $Z$. Moreover, as $\tau$ is fixed on some neighborhood of $Z$, it also fixed near $Z \cap Y_{m-1}$. Therefore $g = h = \tau^{-1} \circ h$ near $Z \cap Y_{m-1}$, and so $g$ is a well defined $C^\infty$ map. It remains to prove that $g^m = \mathbf{F}_\beta \in \mathcal{S}_{id}(f)$ for some $C^\infty$ function $\beta : M \to \mathbb{R}$.

Let $\mathbf{F}$ be a Hamiltonian flow for $f$. Since $\tau$ and $h^m$ are isotopic in $\mathcal{S}(f)$, it follows that $\tau^{-1} \circ h^m \in \mathcal{S}_{id}(f)$. Hence by Lemma 2.3.5, $\tau^{-1} \circ h^m = \mathbf{F}_\alpha$ for some $C^\infty$ function $\alpha : M \to \mathbb{R}$.

Since $\mathcal{S}_{id}(f)$ is a normal subgroup of $\mathcal{S}(f)$, it follows that

\[ h^j \circ (\tau^{-1} \circ h^m) \circ h^{-j} = h^j \circ \tau^{-1} \circ h^{m-j} \in \mathcal{S}_{id}(f), \quad j = 0, \ldots, m-1, \]

as well. Therefore, again by Lemma 2.3.5, $h^j \circ \tau^{-1} \circ h^{m-j} = \mathbf{F}_{\alpha_j}$ for some $C^\infty$ function $\alpha_j : M \to \mathbb{R}$. 
As $\tau$ is fixed on some neighborhood of $Z$, it follows that for each $j$

$$F_{\alpha_j} = h^j \circ \tau^{-1} \circ h^{m-j} = \tau^{-1} \circ h^m = F_{\alpha} \text{ on } Z.$$ 

Then the assumption (1) that every connected component $Z'$ of $Z$ contains a saddle point, implies that $F$ has a non-closed orbit $\gamma$ in $Z'$. Therefore $\alpha = \alpha_j$ on $\gamma$. Since $Z'$ is connected, it follows from local uniqueness of shift functions for $\tau^{-1} \circ h^m|_{Z}$ (see Corollary 4.5) that $\alpha = \alpha_j$ near $Z'$ for all $j = 0, \ldots, m - 1$. Hence $\alpha = \alpha_j$ near all of $Z$ for all $j = 0, \ldots, m - 1$.

Thus we obtain a well-defined $C^\infty$ function $\beta : M \to \mathbb{R}$ given by:

$$\beta(x) = \begin{cases} 
\alpha(x), & x \in Z, \\
\alpha_j(x), & x \in Y_j, \ j = 0, \ldots, m - 1.
\end{cases}$$

We claim that $g^m = F_\beta$.

a) Indeed, if $x \in Z$, then $g^m(x) = h^m(x) = F_{\alpha}(x) = F_{\beta}(x)$.

b) Also notice that $g(Y_i,j) = Y_{i,j+1 \mod m}$ and $g(Y_j) = Y_{j+1 \mod m}$. Then $g^m|_{Y_j}$ is the following composition of maps:

$$Y_j \xrightarrow{h} Y_{j+1} \xrightarrow{h} \cdots \xrightarrow{h} Y_{m-1} \xrightarrow{\tau^{-1} \circ h} Y_0 \xrightarrow{h} \cdots \xrightarrow{h} Y_j,$$

which thus coincides with $h^j \circ \tau^{-1} \circ h^{m-j} = F_{\alpha_j} = F_{\beta}$. \hfill \Box

10. PROOF OF THEOREM 3.6

Let $M$ be a compact surface, $f \in \mathcal{F}(M, P)$, $h \in \mathcal{S}(f)$, $A$ be the union of all regular leaves of $f$ being $h^-$-invariant and $K_1, \ldots, K_k$ be all the critical leaves of $f$ such that $\partial A \cap K_i \neq \emptyset$. For $i = 1, \ldots, k$, let $R_{K_i}$ be an $f$-regular neighborhood of $K_i$ chosen so that $R_{K_i} \cap R_{K_j} = \emptyset$ for $i \neq j$ and

$$Z := A \bigcup \left( \bigcup_{i=1}^{k} R_{K_i} \right).$$

Assume that $Z$ is non-empty, orientable and every connected component $\gamma$ of $\partial Z \cap \text{Int } M$ separates $M$. We have to prove that there exists $g \in \mathcal{S}(f)$ which coincide with $h$ on $Z$ and such that $g^2 \in \mathcal{S}(f)$.

**Lemma 10.1.** There exists a unique $C^\infty$ function $\alpha : Z \to \mathbb{R}$ such that $h^2|_{Z} = F_{\alpha}$ and $\alpha = 0$ at each fixed point of $h^2$ on $h^-$-invariant regular leaves of $f$.

**Proof.** Let $V$ be a regular leaf $V$ of $f$, and $\mathcal{N}$ a chipped cylinder of $f$ such that $V \subset \text{Int } \mathcal{N}$. If $V$ is $h^-$-invariant, then so is every other regular leaf $V' \subset \text{Int } \mathcal{N}$. This implies that $Z$ is a union of $f$-regular neighborhoods $Z_1, \ldots, Z_l$ of some chipped cylinders $\mathcal{N}_1, \ldots, \mathcal{N}_l$ of $f$. 
By Theorem 8.3, for each $i = 1, \ldots, l$ there exists a unique $C^\infty$ function $\alpha_i : Z_i \to \mathbb{R}$ such that $h^2|_{Z_i} = F_{\alpha_i}$ and $\alpha = 0$ at each fixed point of $h^2$ on each on $h^-$-invariant regular leaf of $f$ in $Z_i$.

Notice that if $Z_i \cap Z_j \neq \emptyset$, then every connected component $W$ of that intersection always contains a non-periodic orbit $\gamma$ of $F$. Therefore by uniqueness of shift functions (Corollary 4.5) we obtain that $\alpha_i = \alpha_j$ on $W$.

Hence the functions $\{\alpha_i\}_{i=1, \ldots, l}$ agree on the corresponding intersections, and therefore they define a unique $C^\infty$ function $\alpha : Z \to \mathbb{R}$ such that $h^2|_{Z} = F_{\alpha}$. Then $\alpha = 0$ at each on $h^-$-invariant regular leaf of $f$ in $Z_i$. □

If $M = Z$, then theorem is proved. Thus suppose that $M \neq Z$.

**Lemma 10.2.** The number of connected components $\overline{M \setminus Z}$ is even, and they can be enumerated by pairs of numbers:

\[
Y_1,0 \quad Y_2,0 \quad \ldots \quad Y_a,0 \\
Y_1,1 \quad Y_2,1 \quad \ldots \quad Y_{a,1}
\]

for some $a > 1$ so that $h$ exchanges the rows in (10.1), that is $h(Y_{i,0}) = Y_{i,1}$ and $h(Y_{i,1}) = Y_{i,0}$ for each $i$.

**Proof.** Let $Y_1, \ldots, Y_q$ be all the connected components of $\overline{M \setminus Z}$. Denote $\gamma_i := Y_i \cap Z$. Then by condition (B), $\gamma_i$ is a unique common boundary component of $Y_i$ and $Z$.

Since $h(Z) = Z$, it follows that $h$ induces a permutation of connected components of $\{Y_i\}_{i=1, \ldots, q}$. Moreover, by Lemma 3.5.1 $h(\gamma_i) \cap \gamma_i = \emptyset$, whence $h(Y_i) \cap Y_i = \emptyset$ as well. On the other hand, $h^2 = F_{\alpha}$, whence $h^2(\gamma_i) = F_{\alpha}(\gamma_i) = \gamma_i$, and therefore $h^2(Y_i) = Y_i$.

Thus $\{Y_i\}_{i=1, \ldots, q}$ splits into pairs which are exchanged by $h$. □

Now by Lemma 9.1 with $m = 2$ there exists $g$ such that $g = h$ on $Z$ and $g^2 \in S_{id}(f)$. Theorem 3.6 is completed. □

**References**


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Received: September 12, 2020, accepted: November 18, 2020.

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