Centralizers of elements in Lie algebras of vector fields with polynomial coefficients

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring, and $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in $n$ variables. Denote by $W_n = W_n(\mathbb{K})$ the Lie algebra of all $\mathbb{K}$-derivations on $A$ (in case $\mathbb{C}$ it is the Lie algebra of all vector fields on $\mathbb{C}^n$ with polynomial coefficients). For a given $D \in W_n(\mathbb{K})$ the structure of the centralizer $C_{W_n(\mathbb{K})}(D)$ depends on the field of constants $\ker D = \{ \phi \in R \mid D(\phi) = 0 \}$ (here we extend naturally every derivation $D$ of $A$ on the field $R$). The case $\text{tr. deg}_K \ker D \leq 1$ is studied, the structure of the subalgebra $C_{W_n(\mathbb{K})}(D)$ is characterized, in particular it is proved that if $\ker D$ does not contain any non-constant polynomial, then $C_{W_n(\mathbb{K})}(D)$ is finite-dimensional over $\mathbb{K}$.

Some results about centralizers of linear derivations in $W_n(\mathbb{K})$ are obtained.

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Let $\mathbb{K}$ be an algebraically closed field of characteristic zero (without loss of generality one can assume that $\mathbb{K} = \mathbb{C}$, the field of complex numbers). Denote by $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring and by $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in $n$ variables.

Recall that a \(\mathbb{K}\)-linear map $D : A \to A$ is a \(\mathbb{K}\)-derivation (or simply a derivation) whenever
\[
D(fg) = D(f)g + fD(g)
\]
for all $f, g \in A$. In case $\mathbb{K} = \mathbb{C}$ every \(\mathbb{C}\)-derivation can be considered as a vector field on on \(\mathbb{C}^n\) with polynomial coefficients. We will use this standard correspondence between (polynomial) vector fields and derivations on (polynomial) rings. Any derivation $D$ on $A = \mathbb{K}[x_1, \ldots, x_n]$ can be uniquely extended to the derivation $D$ on $R = \mathbb{K}(x_1, \ldots, x_n)$ (we use the same notation here) by the rule
\[
D(f/g) = (D(f)g - fD(g))/g^2
\]
for all $f, g \in A$, $g \neq 0$.

The Lie algebra $W_n(\mathbb{K})$ of all \(\mathbb{K}\)-derivations on $A$ is of great interest because its finite dimensional subalgebras are closely connected with symmetries of differential equations (recall that any derivation $D$ on $A$ is of the form
\[
D = f_1(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} + \ldots + f_n(x_1, \ldots, x_n) \frac{\partial}{\partial x_n}
\]
for some $f_i \in \mathbb{K}[x_1, \ldots, x_n]$, where $\frac{\partial}{\partial x_i}$ are partial derivatives on $A$).

Finite dimensional subalgebras of the Lie algebras $W_1(\mathbb{C})$ and $W_2(\mathbb{C})$ were classified by S. Lie [4] (more precisely Lie algebras of vector fields with
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analytical coefficients were described in [4]). Analogous problem for \( W_3(\mathbb{C}) \) is open, the problem of classifying all finite-dimensional Lie subalgebras of vector fields from \( W_n(\mathbb{C}), n \geq 4 \) is wild [1].

If \( D \in W_n(\mathbb{K}) \), then the centralizer \( C_{W_n(\mathbb{K})}(D) \) is a subalgebra of \( W_n(\mathbb{K}) \) consisting of all vector fields commuting with \( D \). An information about \( C_{W_n(\mathbb{K})}(D) \) can be useful in many cases. For example, every vector field \( D \in W_n(\mathbb{C}) \), \( D = \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} \) defines an autonomous system of ODE:

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\
\vdots \\
\frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n)
\end{align*}
\]  

(1.1)

with polynomial coefficients and information about \( \ker D \) and \( C_{W_n(\mathbb{K})}(D) \) can be very useful for searching solutions of (1.1) see, for example [5].

Given \( k \) commuting linearly independent over \( R \) vector fields on a smooth \( n \)-manifold \( M \), one can construct a local coordinate system on \( M \) in which these vector fields are of the form \( \frac{\partial}{\partial x_i}, i = 1, \ldots, k \) (see, e.g. [3, Th. 9.46]). We study centralizers of elements \( D \in W_n(\mathbb{K}) \) in case when \( \ker D \) (in the field \( R = \mathbb{K}(x_1, \ldots, x_n) \)) is of transcendence degree \( \leq 1 \) over \( \mathbb{K} \), i.e. any two rational functions \( f, g \) annihilated by \( D \) are algebraically dependent over \( \mathbb{K} \).

In case \( \text{tr.deg}_{\mathbb{K}} \ker D = 0 \) we have \( \ker D = \mathbb{K} \) and then \( C_{W_n}(D) \) is a vector space of dimension \( \leq n \) over \( \mathbb{K} \).

If \( \text{tr.deg}_{\mathbb{K}} \ker D = 1 \), then by Gordan’s theorem (see, e.g. [8]) either \( \ker D = \mathbb{K}(p) \) or \( \ker D = \mathbb{K}(\frac{p}{q}) \), where \( p, q \) are irreducible polynomials that are algebraically independent over \( \mathbb{K} \).

If \( \ker D = \mathbb{K}(p) \), then the centralizer \( C \) is a module over the ring \( \mathbb{K}[p] \) of rank \( k \), \( 1 \leq k \leq n \) and \( C \) is either a Lie algebra over \( \mathbb{K}[p] \) or it contains an ideal \( I \) of rank \( k - 1 \) which is a Lie algebra over \( \mathbb{K}[p] \) and \( C = I + \mathbb{K}[p]T \) for some derivation \( T \in C \) (Theorem 3.1).

In case \( \ker D = \mathbb{K}(p/q) \) we have that

\[
C = (\mathbb{K}(p/q)D + \ldots + \mathbb{K}(p/q)D_{k-1}) \cap W_n(\mathbb{K})
\]

and \( C \) is finite-dimensional over \( \mathbb{K} \) (Theorem 3.3).

We use standard notation. Every derivation \( D \in W_n(\mathbb{K}) \) can be uniquely written in the form

\[
D = f_1(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} + \ldots + f_n(x_1, \ldots, x_n) \frac{\partial}{\partial x_n}
\]

for some \( f_i \in A \). One can show that every nonzero derivation \( D \) can be written in the form \( D = hD_0 \), where \( D_0 \) is reduced, i.e. if \( D_0 = h_1D_1 \) for some \( D_1 \in W_n(\mathbb{K}) \) and \( h_1 \in A \) then \( h_1 \in \mathbb{K}^* \). Denote by \( \overline{W}_n(\mathbb{K}) \) the Lie
algebra of all $\mathbb{K}$-derivations of the field $R = \mathbb{K}(x_1, \ldots, x_n)$. It is obvious that $\widetilde{W}_n(\mathbb{K})$ is a vector space of dimension $n$ over $R$ (with the standard basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$) but not a Lie algebra over $R$.

A rational function $\varphi \in R = \mathbb{K}(x_1, \ldots, x_n)$ is called closed if the subfield $\mathbb{K}(\varphi)$ is algebraically closed in the field $R$.

2. PRELIMINARY RESULTS ABOUT CENTRALIZERS

Lemma 2.1. Let $D \in W_n(\mathbb{K}) \setminus \{0\}$, $F$ be the field of constants of $D$ in $R$, and $C = C_{\widetilde{W}_n(\mathbb{K})}(D)$. Then either

- $C = C_{\widetilde{W}_n(\mathbb{K})}(D) = FD$, or
- $C = FD + FD_2 + \cdots + FD_k$ for some $D_2, \ldots, D_k \in C$ with $D, D_2, \ldots, D_k$ linearly independent over $R$.

Proof. Note that $C$ is a subalgebra of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ over the field $\mathbb{K}$ and $D \in C$. Choose a basis $D, D_1, \ldots, D_k$ (this includes the case $k = 0$) for the vector space $RC$ over the field $R$. Every $T \in C$ (note that $C \subseteq RC$) can be written in the form $T = rD + r_2D_2 + \cdots + r_kD_k$ for some $r, r_i \in R$. But then the equality $[D, T] = 0$ implies $D(r_i) = 0$, $i = 1, \ldots, k$, i.e. $r_i \in \ker D = F$.

On the contrary, one can note that any element from $FD + \cdots + FD_k$ belongs to $C$. Therefore $C = FD + \cdots + FD_k$. If $F \subseteq \ker D_i$ for all $i \geq 2$, then $C$ is not only $k$-dimensional vector space over $F$ but also a Lie algebra over $F$. $\square$

Corollary 2.2. Under assumption of Lemma 2.1, if $F = \mathbb{K}$ then $C$ is a $k$-dimensional Lie algebra over the field $\mathbb{K}$.

Example 2.3. Let $D \in W_n(\mathbb{K})$ be a linear derivation,

$$D = f_1(x_1, \ldots, x_n) \frac{\partial}{\partial x_1} + \cdots + f_n(x_1, \ldots, x_n) \frac{\partial}{\partial x_n},$$

$$f_i(x_1, \ldots, x_n) = D(x_i) = \sum_{j=1}^{n} a_{ij}x_j.$$ Assume also that the Jordan normal form of the matrix $(a_{ij})$ is diagonal

$$\begin{bmatrix}
\lambda_1 & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \lambda_n
\end{bmatrix},$$

where $\lambda_i$ are linearly independent over $\mathbb{Z}$ eigenvalues of the matrix $(a_{ij})$. Then $C = C_{\widetilde{W}_n(\mathbb{K})}(D)$ is of rank $n$ over $R$ and has dimension $n$ over $\mathbb{K}$. Indeed, $\ker D = \mathbb{K}$ by [6, Theorem 10.1.2]. Let

$$L = \left\{ \sum_{j=1}^{n} \mu_jx_j \frac{\partial}{\partial x_j} \mid \mu_j \in \mathbb{K} \right\}.$$
One can easily see that \( L \subseteq C \) and \( \text{rk}_RL = n \). Therefore \( \text{rk}_RC = n \) and \( \dim_{\mathbb{K}}C = n \) by Lemma 2.1.

**Lemma 2.4.** Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero and \( L \) an algebraically closed subfield of the field \( R = \mathbb{K}(x_1, \ldots, x_n) \) with \( \text{tr.deg}_\mathbb{K}L = 1 \). If \( L \) contains a non-constant polynomial from

\[
A = \mathbb{K}[x_1, \ldots, x_n] \subset R,
\]

then \( L = \mathbb{K}(p) \) for some irreducible polynomial \( p \in A \). If \( L \cap A = \mathbb{K} \), then \( L = \mathbb{K}(p/q) \) for some irreducible polynomials \( p, q \in A \) which are algebraically independent over \( \mathbb{K} \).

**Proof.** By Gordan’s theorem (e.g. [8, Theorem 3]) we have that \( L = \mathbb{K}(\varphi) \) for some rational function \( \varphi \in R \).

First let \( L \cap A = \mathbb{K} \). The by [7, Corollary 1], we have that \( L = \mathbb{K}(\frac{p}{q}) \) for some irreducible polynomials \( p, q \) which are algebraically independent over \( \mathbb{K} \). Now let \( L \cap A \neq \mathbb{K} \) and \( r \in (L \cap A)\backslash \mathbb{K} \). Then \( r = F(\frac{p}{q}) \) or \( r = F(p) \) for some rational function \( F(t) \in \mathbb{K}(t) \):

\[
F(t) = \frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n},
\]

with \( a_i, b_j \in \mathbb{K} \), \( a_0b_0 \neq 0 \). If \( r = F(p/q) \), then

\[
F(p/q) = \frac{a_0p^m + \cdots + a_mq^m}{b_0p^n + \cdots + b_nq^n}q^{n-m}
\]

and the numerator and denominator here are homogeneous polynomials in \( p \) and \( q \) of degree \( \max\{m, n\} \). For simplicity assume that \( n \geq m \) for simplicity. Then

\[
r = F(p/q) = \frac{(\alpha_1p + \beta_1q)\cdots(\alpha_np + \beta_nq)}{(\gamma_1p + \delta_1q)\cdots(\gamma_np + \delta_nq)} \tag{2.1}
\]

for some \( \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{K} \) since the ground field \( \mathbb{K} \) is algebraically closed.

Note that the polynomials \( \alpha_ip + \beta_iq \) and \( \gamma_ip + \delta_iq \) are either coprime (when \( \left| \begin{array}{l} \alpha_i \\ \gamma_i \end{array} \right| \beta_i \left| \begin{array}{l} \alpha_i \\ \gamma_i \end{array} \delta_i \right| \neq 0 \)) or proportional with a multiplier in \( \mathbb{K}^* \) when \( \left| \begin{array}{l} \alpha_i \\ \gamma_i \end{array} \right| \beta_i \left| \begin{array}{l} \alpha_i \\ \gamma_i \end{array} \delta_i \right| = 0 \).

Since the rational function \( F(t) \) can be choosen irreducible, the equality (2.1) is impossible because its numerator and denominator are coprime and \( r \) is a non-constant polynomial. Thus the case \( L = \mathbb{K}(p/q) \) is impossible and \( L = \mathbb{K}(p) \) for an irreducible polynomial \( p(x_1, \ldots, x_n) \).

**Proposition 2.5.** Let \( D_1 \in \widetilde{W}_n(\mathbb{K}) \) be such a derivation of the field \( R \) that \( F = \text{ker}D_1 \) in \( R \) is of transcendence degree 1 over \( \mathbb{K} \). Then the centralizer
\[ C = C_{W_n(K)}(D_1) \text{ is a subalgebra of } \tilde{W}_n(K) \text{ of } \text{rk}_R C = k, \ 1 \leq k \leq n, \text{ and} \]
\[ C = FD_1 + FD_2 + \cdots + FD_k \]

for some \( D_2, \ldots, D_k \in C \). Moreover, either \( C \) is a Lie algebra over \( F \) of dimension \( k \), or \( C \) contains an ideal of corank one over \( R \) which is a Lie algebra over \( F \) of dimension \( k - 1 \).

**Proof.** By Gordan’s theorem (e.g. [8, Theorem 3]) we have that \( F = K(\varphi) \) for some closed rational function \( \varphi \in R \). Choose a basis \( D_1, D_2, \ldots, D_k \) of \( C \) over \( R \). As \( [D_1, D_i] = 0, i = 1, \ldots, k \), we have \( D_i(\ker D_1) \subseteq \ker D_1 \). So \( D_i(\varphi) = f_i(\varphi) \) for some rational functions \( f_i(t), i = 1, \ldots, k \).

If \( f_1(t) = \cdots = f_n(t) = 0 \), then \( F \subseteq \ker D_i \) for \( i = 1, \ldots, k - 1 \). Therefore, \( C = FD_1 + \cdots + FD_k \) is a \( k \)-dimensional Lie algebra over the field \( F \).

Now suppose that \( f_i(t) \neq 0 \) for some \( i, 2 \leq i \leq n \). Then one can easily prove that \( f_i(\varphi) \neq 0 \). Denote by \( C_0 = \{ T \in C \mid T(\varphi) = 0 \} \) the annihilator of the element \( \varphi \) in \( C \). Since \( D_i(\ker D) \subseteq \ker D \), we see that \( C_0 \) is an ideal of \( C \). We claim that \( \text{rk}_R C_0 = k - 1 \). Indeed, if \( T, S \in C \setminus C_0 \) then \( T(\varphi) = g(\varphi) \) and \( S(\varphi) = h(\varphi) \) for some nonzero rational functions \( g(t) \) and \( h(t) \). It now follows that \( h(\varphi)T - g(\varphi)S \in C_0 \) and therefore \( \text{rk}_R C/C_0 = 1 \). Thus we have \( \text{rk}_R C_0 = k - 1 \).

Next, we point out a series \( D_2, \ldots, D_n \) of derivations on the polynomial ring \( K[x_1, \ldots, x_n] \) with centralizers \( C_i = C_{W_n}(D_i) \) such that
\[ \text{rank}_R C_i = n - i + 1, \quad i = 2, \ldots, n. \]
We use the known simple derivation from [6, Example 13.4.3].

**Lemma 2.6.** Let \( D_k = \frac{\partial}{\partial x_1} + (1 + x_1 x_2) \frac{\partial}{\partial x_2} + \cdots + (1 + x_{k-1} x_k) \frac{\partial}{\partial x_k} \) be a derivation of the polynomial ring \( K[x_1, \ldots, x_n], 2 \leq k \leq n \). Then
(1) \( \ker D_k = K[x_{k+1}, \ldots, x_n] \) for \( k < n \) and \( \ker D_n = K; \)
(2) \( C_k = C_{W_n(K)}(D_k) = K[x_{k+1}, \ldots, x_n] D_k + K[x_{k+1}, \ldots, x_n] \frac{\partial}{\partial x_{k+1}} + \cdots + K[x_{k+1}, \ldots, x_n] \frac{\partial}{\partial x_n} \),

for \( k < n \) and \( C_n = K D_n \).

In particular, \( \text{rk}_R(C_k) = n - k + 1 \).

**Proof.** (1) The polynomial ring \( A = K[x_1, \ldots, x_n] \) can be considered as the polynomial ring in variables \( x_1, \ldots, x_k \) over the ring \( F := K[x_{k+1}, \ldots, x_n] \).

By [6, Example 13.4.3] \( D_k \) is a simple derivation of the ring \( F[x_1, \ldots, x_k] \) (note that \( F \subseteq \ker D_k \)). Hence the kernel of \( D_k \) in \( F[x_1, \ldots, x_k] \) coincides with \( F \). Therefore the kernel of the derivation \( D_k \) in the ring \( A \) coincides with \( K[x_{k+1}, \ldots, x_n] \).
(2) Let \( T \in C = C_{W_n}(\mathbb{K}) \),

\[
T = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}.
\]

Then the equality \([T, D_k] = 0\) implies equalities \( D_k(f_1) = T(1) = 0 \) and therefore \( f_1 \in \mathbb{K}[x_{k+1}, \ldots, x_n] \),

\[
D_k(f_2) = x_1 f_2 + x_2 f_1, \quad \ldots, \quad D_k(f_{k}) = x_{k-1} f_k + x_k f_{k-1},
\]

\[
D_k(f_{k+1}) = 0, \quad \ldots, \quad D_k(f_n) = 0.
\]

The last \( n-k \) equalities imply that

\[
f_{k+1} \in \mathbb{K}[x_{k+1}, \ldots, x_n], \quad \ldots, \quad f_n \in \mathbb{K}[x_{k+1}, \ldots, x_n].
\]

If \( f_1 \neq 0 \), then \( f_1 D_k \in C_k \) and \( T - f_1 D_k \in C_k \). Therefore without loss of generality one can assume that \( f_1 = 0 \). But then \( D_k(f_2) = x_1 f_2 \) which is possible only if \( f_2 = 0 \) because \( D_k \) is a simple derivation of the ring \( F[x_1, \ldots, x_k] \). Repeating the arguments one can conclude that

\[
f_3 = \cdots = f_k = 0.
\]

The latter means that

\[
T - f_1 D_k \in \mathbb{K}[x_{k+1}, \ldots, x_n] \frac{\partial}{\partial x_{k+1}} + \cdots + \mathbb{K}[x_{k+1}, \ldots, x_n] \frac{\partial}{\partial x_n}.
\]

Taking into account the relation \( f_1 \in \mathbb{K}[x_{k+1}, \ldots, x_n] \) we get the needed statement. \( \square \)

In order to separate factors of a polynomial which belong to the kernel of a derivation we consider the following notions. Let \( p \in \mathbb{K}[x_1, \ldots, x_n] \) be an irreducible polynomial. A polynomial \( f = f(x_1, \ldots, x_n) \) will be called \( p\)-free if \( f \) is not divisible by any polynomial in \( p \) of positive degree. It can be easily shown that every polynomial \( g \in \mathbb{K}[x_1, \ldots, x_n] \) can be written in the form \( g = g_0 g_1 \), where \( g_0 \) is a \( p \)-free polynomial and \( g_1 = g_1(p) \) is a polynomial of \( p \) (this includes the case \( g_1 = \text{const} \)). The degree in \( p \) of the polynomial \( g_1(p) \) will be called the \( p\)-degree of \( g \) and denoted by \( \deg_p g \).

Let \( p \) and \( q \) be algebraically independent irreducible polynomials of the ring \( \mathbb{K}[x_1, \ldots, x_n] \). A polynomial \( f(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n] \) will be called \( p-q\)-free if \( f \) is not divisible by any homogeneous polynomial in \( p \) and \( q \) of positive degree. As earlier one can write every polynomial \( g \in \mathbb{K}[x_1, \ldots, x_n] \) in the form \( g_0 g_1 \), where \( g_0 \) is a \( p-q\)-free polynomial and \( g_1 \) is a homogeneous polynomial in \( p, q \). The (total) degree of \( g_1 \) in \( p, q \) will be called the \( p-q\)-degree of \( g \) and denoted by \( \deg_{p-q} g \).

If \( D \) is a derivation on the polynomial ring \( \mathbb{K}[x_1, \ldots, x_n] \), then \( D \) can be written in the form \( h D_0 \), where \( D_0 \) is an irreducible derivation on \( \mathbb{K}[x_1, \ldots, x_n] \) and \( h \in \mathbb{K}[x_1, \ldots, x_n] \). We will call \( D \) \( p\)-free if the polynomial \( h \) is \( p\)-free. We summarize all these remarks in the next statement.
Lemma 2.7. Let $D \in W_n(\mathbb{K})$ be a nonzero derivation. Then there exist unique (up to a factor from $\mathbb{K}^*$) polynomials $f(p, q)$ and $h$ such that

$$D = f(p, q)hD_0,$$

where $D_0$ is a reduced derivation, $f(p, q)$ is a homogeneous polynomial in $p, q$ and the polynomial $h$ is $p$-$q$-free.

3. CENTRALIZERS OF ELEMENTS IN $W_n(\mathbb{K})$

Theorem 3.1. Let $D$ be a derivation of the ring $\mathbb{K}[x_1, \ldots, x_n]$ with the field of constants $F = \ker D$ in $R = \mathbb{K}(x_1, \ldots, x_n)$ of the form $F = \mathbb{K}(p)$ for some irreducible polynomial $p$ and let $C = C_{W_n(\mathbb{K})}(D)$. Then

1. If $\text{rk}_R C = 1$, then $C = \mathbb{K}[p]D_0$ for some $p$-free derivation $D_0$ with $D = f(p)D_0$ for some $f(t) \in \mathbb{K}[t]$;

2. If $\text{rk}_R C \geq 2$, then $C$ is either a Lie algebra of rank $k$ over the ring $\mathbb{K}[p]$ or $C$ contains an ideal $I$ of rank $k-1$ that is a Lie algebra over $\mathbb{K}[p]$ and $C = I + \mathbb{K}[p]S$ for some derivation $S \in C$.

Proof. As noted above the derivation $D$ can be written in the form

$$D = f(p)D_0,$$

where $D_0$ is a $p$-free derivation and the polynomial $f \in \mathbb{K}[t]$ is uniquely defined by $D$ up to a nonzero multiplier in $\mathbb{K}^*$.

1. First let $\text{rk}_R C = 1$. Take an arbitrary element $T \in C$. Then $T = \varphi(p)D_0$ for some rational function $\varphi \in \mathbb{K}(t)$, where $\varphi(p) = g(p)/h(p)$ for some polynomials $g(t), h(t) \in \mathbb{K}[t]$. Without loss of generality one can assume that $\varphi(t) = g(t)/h(t)$ is a reduced fraction. It follows from the equality $T = \varphi(p)D_0$ that $h(p)T = g(p)D_0$. Write $D_0$ and $T$ in the form

$$D_0 = \sum_{i=1}^n P_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \quad T = \sum_{j=1}^n Q_j(x_1, \ldots, x_n) \frac{\partial}{\partial x_j},$$

where $P_i, Q_j \in \mathbb{K}[x_1, \ldots, x_n]$. Suppose that the polynomial $h$ is non-constant. Since $D_0$ is $p$-free, at least one of the coefficients of $D_0$ is not a multiple of $h(p)$. Without loss of generality one can assume that $P_1$ is such a coefficient. Then it follows from the equality $h(p)T = g(p)D_0$ that $hQ_1 = gP_1$. Taking into account the equality $(g(p), h(p)) = 1$ we see that $h|P_1$ which gives a contradiction. Therefore $h \in \mathbb{K}^*$ and

$$\varphi = g(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n].$$

But then $T = g(p)D_0$ and $C = \mathbb{K}[p]D_0$ since $T$ was arbitrarily chosen.

2. Let $\text{rk}_R C = k \geq 2$. If for each $D_1 \in C$ we have that $D_1(F) = 0$, then it is easy to see that $C$ is a Lie algebra of rank $k$ over the ring $\mathbb{K}[p]$. Note that in this case $C$ may not be a free $\mathbb{K}[p]$-module.
Suppose there exists an element $S \in C$ such that $S(F) \neq 0$. Then $S(p) \neq 0$. Choose $S$ so that the $p$-degree of the polynomial $S(p)$ is minimal.

We claim that for each $T \in C$ the polynomial $T(p)$ is a multiple of $S(p)$. Indeed suppose $S(p) = v(p)$, $T(p) = u(p)$ for some polynomials $v(t), u(t) \in \mathbb{K}[t]$. Write $u(t) = v(t)q(t) + r(t)$ for some polynomials $q(t), r(t)$, where $\deg r(t) < \deg v(t)$. Then $u(p) = v(p)q(p) + r(p)$ and $T - q(p)S \in C$. Since $(T - q(p)S)(p) = r(p)$ and $\deg p r(p) < \deg p S(p)$, we have by the choice of $S$ that $r(p) = 0$ and $T - q(p)S$ annihilates the kernel $\ker D$. Denote by $C_0$ the subalgebra of $C$ of all derivations annihilating $\mathbb{K}[p]$. Then as was shown above $T - q(p)S \in C_0$ and $C = C_0 + \mathbb{K}[p]S$. □

**Corollary 3.2.** If $k = 2$ and $C(F) \neq 0$, then $C = \mathbb{K}[p]D_0 + \mathbb{K}[p]S$ is a free module of rank 2 over $\mathbb{K}[p]$. 

**Theorem 3.3.** Let $D \in W_n(\mathbb{K})$ be a derivation with

$$\text{tr. deg}_{\mathbb{K}} \ker D = 1 \quad \text{and} \quad (\ker R D) \cap A = \mathbb{K}.$$ 

Then

1) $\ker D = \mathbb{K}(p/q)$ for some irreducible algebraically independent polynomials $p, q \in \mathbb{K}[x_1, \ldots, x_n],$

2) the derivation $D$ is of the form $D = hf(p,q)D_0$ for some irreducible derivation $D_0$ and homogeneous in $p,q$ polynomial $f$ and a $p$-$q$-free polynomial $h,$

3) the centralizer $C = C_{W_n(\mathbb{K})}(D)$ is finite-dimensional over $\mathbb{K}$ being one of the following types:

   (a) $C = \mathbb{K}[p,q]_m hD_0$, where $\mathbb{K}[p,q]_m$ is the linear space of homogeneous in $p,q$ polynomials of degree $m = \deg p,q f$, and in particular $\dim_{\mathbb{K}} C = m + 1$;

   (b) $C = (\mathbb{K}(p/q)D + \mathbb{K}(p/q)D_2 + \cdots + \mathbb{K}(p/q)D_k) \cap W_n(\mathbb{K})$ for some elements $D_2, \ldots, D_k$, $k \leq n$ in $C$ with $D, D_2, \ldots, D_k$ linearly independent over the field $R$.

**Proof.** By Lemma 2.4 we have that $\ker D = \mathbb{K}(p/q)$ for some irreducible algebraically independent over $\mathbb{K}$ polynomials $p, q$. By Lemma 2.7 there exist unique (up to a nonzero factor from $\mathbb{K}$) polynomials $f(p,q)$ and $h$ such that $D = f(p,q)hD_0$, where $D_0$ is a reduced derivation, $f(p,q)$ is a homogeneous polynomial at $p,q$ and the polynomial $h$ is $p$-$q$-free.

First, let $\text{rk}_R C = 1$. Then for any $D_1 \in C$ we have that $D_1 = rD_0$ for some $r \in A$ (because $D_0$ is a reduced derivation). As mentioned above, $r = f_1h_1$ for some homogeneous polynomial $f_1(p,q)$ in $p,q$ and a $p$-$q$-free
polynomial $h_1$. By the choice of $D_1$ we have that
\[ 0 = [D, D_1] = [f_1 h_1 D_0, f h D_0]. \]
The last relation implies the equality
\[ D_0 (f h / (f_1 h_1)) = 0. \]
By Lemma 2.4 $f h / (f_1 h_1) = u(p, q) / v(p, q)$ for some homogeneous polynomials $u, v$ in $p, q$ with $\deg u = \deg v$. Hence
\[ h f v = h_1 f_1 u, \]
where $f v$ and $f_1 u$ are homogeneous in $p, q$ and $h, h_1$ are $p$-$q$-free polynomials. Recall that the factorization of a polynomial as a product of a homogeneous in $p, q$ and a $p$-$q$-free polynomial is unique up to a factor from $\mathbb{K}^*$. Hence $h_1 = h c, c \in \mathbb{K}^*$, and $f v = c^{-1} f_1 u$. Then
\[ \deg_{p-q} f = \deg_{p-q} f_1 = m. \]
This implies the relation
\[ D = f_1 h_1 D_0 \in \mathbb{K}[p, q]_{m} h D_0. \]
Since $D_1$ was arbitrarily chosen in $C$, we have the inclusion
\[ C \subseteq \mathbb{K}[p, q]_{m} h D_0. \]
It is easy to see that $\mathbb{K}[p, q]_{m} h D_0 \subseteq C$ and therefore $C = \mathbb{K}[p, q]_{m} h D_0$.

Further, let $\text{rk}_R C = k \geq 2$. Choose a basis $D, D_2, \ldots, D_k$ of $C$ over $R$. Then by Proposition 2.5
\[ C = (\mathbb{K}(p/q) D + \mathbb{K}(p/q) D_2 + \ldots + \mathbb{K}(p/q) D_k) \cap W_n(\mathbb{K}). \]
We will show by induction on $k$ that the centralizer $C = C_{W_n(\mathbb{K})}(D)$ is finite-dimensional over $\mathbb{K}$.

For $k = 1$ (i.e. in case $\text{rk}_R C = 1$) this was proved above, so we may assume that $k \geq 2$.

Denote for convenience $D_1 = D$. Then every element $D_i$ can be written in the form $D_i = \sum_{j=1}^{n} P_{ij} \frac{\partial}{\partial x_j}$ for some polynomials $P_{ij} \in A, i = 1, \ldots, k$. Take an arbitrary element $T$ of the centralizer $C$ and write down it in the form $T = \sum_{i=1}^{k} \alpha_i D_i$ for some rational functions $\alpha_i \in R$. On the other hand, the same derivation can be written in the standard form $T = \sum_{j=1}^{n} Q_j \frac{\partial}{\partial x_j}$ for some polynomials $Q_1, \ldots, Q_n \in A$. Consider the derivations $D_1, \ldots, D_{k-1}, T$ and denote by $(P'_{ij})$ the polynomial matrix whose first $k - 1$ rows consist of coefficients of derivations $D_1, \ldots, D_{k-1}$ and the $k$-th row is of the form $(Q_1, \ldots, Q_n)$, i.e. $P'_{ij} = P_{ij}$ and $P'_{kj} = Q_j$ for $i = 1, \ldots, k - 1, j = 1, \ldots, n$. 
Consider the minor $\delta = \delta_{i_1, \ldots, i_k}$ on arbitrarily chosen columns $i_1, \ldots, i_k$ of the matrix $(P_{ij})$ and the analogous minor $\mu = \mu_{i_1, \ldots, i_k}$ on the same columns of the matrix $(P'_{ij})$. Since $T = \sum_{i=1}^{k} \alpha_i D_i$, we have obviously the equality $\mu = \alpha_k \delta$.

Repeating the arguments from the proof of Lemma 2.4 one can show that there exist homogeneous polynomials $u, v$ in $p, q$ with $\deg_{p-q} u = \deg_{p-q} v$ such that $\alpha_k = u/v$. It follows from the equality $\mu = \alpha_k \delta$ (written in the form $v\mu = u\delta$) that $\deg_{p-q} \mu = \deg_{p-q} \delta$. Moreover, these polynomials have the same $p-q$-free part up to a factor from $\mathbb{K}^*$ because of the equality $v\mu = u\delta$ mentioned above. We can assume that $p-q$-free parts of $u$ and $v$ are identical, denote their common value by $h$. Let $M_1, \ldots, M_s$ be all the $(k \times k)$-minors of the matrix $(P'_{ij})$ enumerated in an arbitrary way, so $s = \binom{n}{k}$. Then they are polynomials from $A$. Let $m = m_i$ be the $p$-$q$-degree of the minor $M_i$ and $f_i$ the corresponding homogeneous polynomial which is a $p-q$-part of the minor $M_i$. We assign to the derivation $T$ the sequence of homogeneous polynomials $\theta(T) = (f_1, \ldots, f_s)$ of degrees $m_1, \ldots, m_s$ correspondingly. Consider the map

$$\theta : C \rightarrow N = \mathbb{K}[p, q]_{m_1} \times \ldots \times \mathbb{K}[p, q]_{m_s},$$

where $m_i$ are $p$-$q$-degree of the minor $M_i, i = 1, \ldots, s$. The mapping $\theta$ is $\mathbb{K}$-linear and acts from $C$ to $N$, note that $\dim_{\mathbb{K}} N < \infty$. Obviously $\ker \theta$ consists of such derivations $T$ for which all the minors of order $k$ are zeroes. But then

$$T \in (\mathbb{K}(p/q) D_1 + \ldots + \mathbb{K}(p/q) D_{k-1}) \cap W_n(\mathbb{K}) = C_{k-1}.$$

Therefore, $\dim C/C_{k-1} < \infty$. By inductive assumption the subspace $C_{k-1}$ is finite dimensional over the field $\mathbb{K}$. Therefore $\dim_{\mathbb{K}} C < \infty$. □

4. CENTRALIZERS OF SOME LINEAR DERIVATIONS

A derivation $D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial x_i}$ will be called linear if all the polynomial $P_i$ are linear forms in $n$ variables, i.e. $P_i = \sum_{j=1}^{n} a_{ij} x_j, a_{ij} \in \mathbb{K}$. The linear derivation $D = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j}$ is determined by the square matrix $(a_{ij})$ of order $n$ and if $D' = \sum_{i,j=1}^{n} b_{ij} x_j \frac{\partial}{\partial x_j}$, then $[D, D']$ is linear and defined by the matrix $(c_{ij}) = [(a_{ij}), (b_{ij})]$. Therefore all the linear derivation form a subalgebra of $W_n(\mathbb{K})$ isomorphic to the general linear algebra $gl_n(\mathbb{K})$, which (for simplicity) will also be denoted by $gl_n(\mathbb{K})$. 
Let $D = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} \in W_n(\mathbb{K})$ be a linear derivation. Then one can consider two centralizers:

$$C_0 = C_{gl_n(\mathbb{K})}(D) \quad \text{and} \quad C = C_{W_n(\mathbb{K})}(D),$$

Evidently, $C_0 \subseteq C$. The structure of the centralizer $C_0$ is well-known because it consists of all linear derivations defined by the matrices commuting with the matrix $(a_{ij})$. How to find the centralizer of a given matrix $(a_{ij})$ is a classical problem of linear algebra. It was solved many years ago (see, e.g. [2, Chapter VIII, §2]). Therefore it is interesting to study the case when $C = C_0$ because we will then have a complete description of the centralizer $C = C_{W_n(\mathbb{K})}(D)$.

In Theorem 4.2 we will present a necessary condition and a sufficient condition for a linear derivation $D$ to satisfy the equality

$$C_{W_n(\mathbb{K})}(D) = C_{gl_n(\mathbb{K})}(D)$$

(unfortunately those conditions do not coincide).

**Lemma 4.1.** Let $D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ and $T = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$ be two elements of $W_n(\mathbb{K})$, where $f_i = f_i(x_1, \ldots, x_n)$, $g_i = g_i(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$. Then the derivations $D$ and $T$ commute if and only if $D(g_i) = T(f_i)$ for all $i = 1, \ldots, n$.

**Proof.** It is obvious that $DT = TD$ if and only if $DT(x_i) = TD(x_i)$ for all $i = 1, \ldots, n$. But $T(x_i) = g_i(x_1, \ldots, x_n)$ and $D(x_i) = f_i(x_1, \ldots, x_n)$ for all $i = 1, \ldots, n$. Hence $D(g_i) = T(f_i)$. \[\square\]

**Theorem 4.2.** Let $D = \sum_{i,j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_i}$ be a linear derivation of the polynomial ring $K[x_1, \ldots, x_n]$, and $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix $(a_{ij})$. Then the following statements hold:

(1) If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{Z}$, then $C_{W_n(\mathbb{K})}(D) = C_{gl_n(\mathbb{K})}(D)$.

(2) If $C_{W_n(\mathbb{K})}(D) = C_{gl_n(\mathbb{K})}(D)$, then the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{N} \cup \{0\}$.

**Proof.** (1) Suppose that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $(a_{ij})$ are linearly independent over $\mathbb{Z}$. Take any $T \in C_{W_n(\mathbb{K})}(D)$,

$$T = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$
where $f_i \in A = \mathbb{K}[x_1, \ldots, x_n]$. Without loss of generality one may assume that the matrix $(a_{ij})$ is diagonal of the form

$$(a_{ij}) = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}$$

(the eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise distinct, so the matrix $(a_{ij})$ is diagonalizable). In view of this assumption the derivation $D$ is of the form

$$D = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}.$$  

Evidently, $D(f_i) = T(\lambda x_i) = \lambda_i f_i$, i.e. the coefficients $f_i$ of the derivation $T$ are Darboux polynomials for $D$ with cofactors $\lambda_i$, $i = 1, \ldots, n$. Moreover, $D(x_i) = \lambda_i x_i$, $i = 1, \ldots, n$. But then

$$D(f_i/x_i) = \frac{D(f_i)x_i - f_iD(x_i)}{x_i^2} = 0, \quad i = 1, \ldots, n,$$

i.e. the rational function $f_i/x_i$ belongs to the kernel of $D$, $i = 1, \ldots, n$. Since all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{Z}$ it follows from [6, Theorem 10.1.2] that $f_i/x_i = \mu_i \in \mathbb{K}$, $i = 1, \ldots, n$. The latter means that

$$T = \sum_{i=1}^{n} \mu_i x_i \frac{\partial}{\partial x_i} \in gl_n(\mathbb{K})$$

and therefore $C_{W_n(\mathbb{K})}(D) = C_{gl_n(\mathbb{K})}(D)$.

(2) Suppose that

$$C_{W_n(\mathbb{K})}(D) = C_{gl_n(\mathbb{K})}(D).$$

This implies that $\ker D = \mathbb{K}$. Indeed, if $h \in \ker D \setminus \mathbb{K}$, then $hD \in C_{W_n(\mathbb{K})}(D)$ and the derivation $hD$ is obviously nonlinear. Hence by [6, Theorem 10.1.1] the eigenvalues $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{N}_0$. \hfill $\square$

**Remark 4.3.** Note that the derivation

$$D = x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}$$

on the polynomial ring $\mathbb{K}[x_1, x_2]$ with eigenvalues 1, 2 has nonlinear elements in its centralizer in $W_2(\mathbb{K})$, for example $x_1^2 \frac{\partial}{\partial x_2}$. So the condition (2) is not sufficient.
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