Triples of infinite iterations of hyperspaces of max-plus compact convex sets

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Abstract. Geometry of the infinite iterated hyperspace of compact max-plus convex sets, their completions and compactifications is investigated.

1. INTRODUCTION

In their paper [10] H. Toruńczyk and J. West investigated the construction of the iterated hyperspace functor. For a compact metric space $X$, this construction leads to the metric direct limit $X'$ of the sequence

$$X \to \exp X \to \exp^2 X \to \ldots,$$

where every map is the singleton embedding $x \mapsto \{x\}$. In particular, they proved that, for any Peano continuum $X$, the completion $X^*$ of $X'$ is homeomorphic to the separable Hilbert space $l^2$.

The paper [14] is devoted to the construction of iterated superextension (the superextension functor was defined by J. de Groot [3]). It turned out that the completed infinite iterated superextension admits a natural compactification, which is the inverse limit of iterated superextensions. This result was considerably generalized by V. V. Fedorchuk [4]. He introduced the notion of perfectly metrizable functor and described the topology of obtained triples comprised of infinite iterations, their completions, and compactifications by means of inverse systems.

As a partial case, Fedorchuk considered the probability measure functor $P$. The direct and inverse sequences of iterated spaces of probability measures were also considered in [11], [12]. R. Mirzakhanyan [7], [8] investigated the case of the inclusion hyperspace functor.

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The aim of this note is to extend results of [2] onto the case of the so-called max-plus convexity (see the definition below).

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2. Preliminaries

All spaces are assumed to be metrizable topological spaces. Let \((X, d)\) be a metric space. By \(\exp X\) we denote the hyperspace of a space \(X\), i.e., the set of all nonempty compact subsets in \(X\) endowed with the Hausdorff metric \(d_H\):

\[
d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), \ B \subset O_\varepsilon(A)\}.
\]

The Hausdorff metric on \(
\exp^2 X = \exp \exp X\) induced by the (Hausdorff) metric \(d_H\) will be denoted by \(d_{HH}\).

By \(Q = [0, 1]^\omega\) we denote the Hilbert cube. A closed set \(A\) in \(Q\) is called a \(Z\)-set in \(Q\) if the identity map of \(Q\) can be approximated by maps whose images miss \(A\). A subset \(A \subset Q\) is called a \(Z\)-skeletoid [1] if \(A = \bigcup_{i=1}^\infty A_i\), where \(A_1 \subset A_2 \subset \ldots\) is a sequence of \(Z\)-sets satisfying the condition: for each \(\varepsilon > 0\), \(n \in \mathbb{N}\) and a \(Z\)-set \(C \subset Q\) there exist \(m > n\) and an autohomeomorphism \(\psi_\varepsilon: Q \to Q\) such that

1. \(d(\psi_\varepsilon, \text{id}) < \varepsilon\);
2. \(\psi_\varepsilon|_{C \cap A_n} = \text{id}\);
3. \(\psi_\varepsilon(C) \subset A_m\).

(here \(d\) denotes a fixed compatible metric on \(Q\)). See [1] for the necessary properties of \(Z\)-skeletoids in \(Q\).

Recall that a map \(f: X \to Y\) is called soft [9] provided that for every commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow f \\
Z & \xrightarrow{\psi} & Y
\end{array}
\]

such that \(Z\) is a paracompact space and \(A\) is a closed subset of \(Z\) there exists a map \(\Phi: Z \to X\) such that \(f \circ \Phi = \psi\) and \(\Phi|_A = \varphi\).

Let \(\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}\) and let \(\tau\) be a cardinal number. Given \(x, y \in \mathbb{R}^\tau\) and \(\lambda \in \mathbb{R}\), we denote by \(x \oplus y\) the coordinate wise maximum of \(x\) and \(y\) and by \(\lambda \odot x\) the vector obtained from \(x\) by adding \(\lambda\) to every its coordinate. A subset \(A\) in \(\mathbb{R}^\tau\) is said to be max-plus convex if \(\alpha \odot a \oplus \beta \odot b \in A\) for all \(a, b \in A\) and \(\alpha, \beta \in \mathbb{R}_{\max}\) with \(\alpha \oplus \beta = 0\). See, e.g., [6] for the history and applications of max-plus convexity.
A *max-plus convex body* in \( \mathbb{R}^n \) is a max-plus convex set in \( \mathbb{R}^n \) which is the closure of its interior.

The hyperspace of all compact max-plus convex subsets of \( X \subset \mathbb{R}^r \) is denoted by \( \text{mpcc}(X) \).

Remark that there is a natural max-plus (respectively, max-min) convex structure on the hyperspace \( \text{mpcc}(X) \), where \( X \) is a max-plus (respectively max-min) convex compact subset of \( \mathbb{R}^\alpha \), \( 1 \leq \alpha \leq \omega \).

Given a subset \( \mathcal{A} \) of the hyperspace \( \text{mpcc}(X) \), we say that \( \mathcal{A} \) is *max-plus convex* if, for every \( A_1, \ldots, A_n \in \mathcal{A} \) and every \( \alpha_1, \ldots, \alpha_n \in [-\infty, 0] \) with \( \oplus_{i=1}^n \alpha_i = 0 \), we have

\[
\bigoplus_{i=1}^n \alpha_i \circ A_i = \left\{ \bigoplus_{i=1}^n \alpha_i \circ a_i \mid a_i \in A_i, \ i = 1, \ldots, n \right\} \in \mathcal{A}.
\]

Remark that the set \( \bigoplus_{i=1}^n \alpha_i \circ A_i \) is easily seen to be an element of the hyperspace \( \text{mpcc}(X) \). We denote by \( \text{mpcc}^2(X) \) the set of nonempty closed max-plus convex subsets in \( \text{mpcc}(X) \).

One can similarly define the iterations \( \text{mpcc}^m(X) \), \( m \geq 3 \).

## 3. Infinite iterated hyperspaces

Let \( \text{mpcc}^2(X) \) denote the set of all nonempty closed convex subsets in \( \text{mpcc}(X) \), where \( X \) is a compact max-plus convex subspace in \( \mathbb{R}^n \), \( n \geq 1 \).

We endow \( \mathbb{R}^n \) with the \( \ell_\infty \)-metric: if

\[
x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n,
\]

then \( d(x, y) = \max_i |x_i - y_i| \). Note that the union map

\[
u_X : \text{mpcc}^2(X) \to \text{mpcc}(X)
\]

is well-defined. Indeed, if \( \mathcal{A} \in \text{mpcc}(X) \) and for any \( a, b \in u_X(\mathcal{A}) \) there are \( A, B \in \mathcal{A} \) such that \( a \in A \) and \( b \in B \). Since \( \mathcal{A} \) is max-plus convex, for any \( \alpha, \beta \in \mathbb{R}_{\max} \) with \( \alpha \oplus \beta = 0 \) we have \( \alpha \circ A \oplus \beta \circ B \in \mathcal{A} \) and therefore \( \alpha \circ a \oplus \beta \circ b \in u_X(\mathcal{A}) \).

**Lemma 3.1.** For every \( a \in X \) and \( \mathcal{A} \in \text{mpcc}(X) \),

\[
d_{\text{HH}}(\{a\}, \mathcal{A}) = d_H(\{a\}, u_X(\mathcal{A})).
\]

**Proof.** First,

\[
d_{\text{HH}}(\{a\}, \mathcal{A}) = \sup\{d_H(\{a\}, A) \mid A \in u_X(\mathcal{A})\} \leq d_H(\{a\}, u_X(\mathcal{A})).
\]

On the other hand, if \( d_{\text{HH}}(\{a\}, \mathcal{A}) < r \), then \( O_r(a) \supset B \), for every \( B \in \mathcal{A} \).

Therefore, \( O_r(a) \supset u_X(\mathcal{A}) \) and thus \( d_H(\{a\}, u_X(\mathcal{A})) < r \). This proves the reverse inequality. \( \square \)

**Proposition 3.2.** The map \( u_X \) is soft.
Proof. First we show that \( u_X^{-1}(A) \) is max-plus convex for any \( A \in \text{mpcc}(X) \). If \( B, C \in u_X^{-1}(A) \) and \( \beta, \gamma \in \mathbb{R}_{\max} \) with \( \beta + \gamma = 0 \), then we have

\[
\beta \odot B \odot \gamma \odot C = \{ \beta \odot B \odot \gamma \odot C \mid B \in B, \ C \in C \}.
\]

Given \( x \in \beta \odot B \odot \gamma \odot C \in \beta \odot B \odot \gamma \odot C \), we see that there are \( b \in B \) and \( c \in C \) such that \( x = \beta \odot b \odot \gamma \odot c \). Since \( b, c \in A \), we conclude that \( u_X(\beta \odot B \odot \gamma \odot C) \subseteq A \).

Now, if \( x \in A \), then there \( B \in B \) and \( c \in C \) such that \( x \in B \cap C \). Then

\[
x \in \beta \odot B \odot \gamma \odot C \in \beta \odot B \odot \gamma \odot C.
\]

Thus, \( u_X(\beta \odot B \odot \gamma \odot C) \supseteq A \), i.e. finally \( u_X(\beta \odot B \odot \gamma \odot C) = A \).

We are going to prove that the map \( u_X \) is open. Since the spaces under consideration are compact and metrizable, it suffices to prove that for any \( A \in \text{mpcc}^2(X) \) and any sequence \( (A_i) \) in \( \text{mpcc}(X) \) converging to

\[
A = u_X(A)
\]

there exists a sequence \( (A_i) \) in \( \text{mpcc}^2(X) \) converging to \( A \) and such that \( u_X(A_i) = A_i \), for every \( i \in \mathbb{N} \) (see, e.g., [5]).

For any \( i \), let \( r_i = d_H(A, A_i) \) and let

\[
A_i = \overline{\text{conv}}_{\text{mpcc}}(\{A_i \cap O_{r_i}(C) \mid C \in A\})
\]

(by \( \overline{\text{conv}}_{\text{mpcc}} \) we denote the closed max-plus convex hull map). Since the map \( K \mapsto O_{r_i}(K) \) is continuous, we conclude that

\[
\{A_i \cap O_{r_i}(C) \mid C \in A\} \in \exp \text{mpcc}(X).
\]

It is easy to see that \( d_{\HH}([A_i \cap O_{r_i}(C) \mid C \in A], A) \leq r_i \) and, since the closed max-plus convex hull map is nonexpanding, we obtain that \( d_{\HH}(A_i, A) \leq r_i \).

Now, by [13, Theorem 3.3] the map \( u_X \) is soft as an open map with max-plus convex preimages. \( \square \)

Given a compact convex set \( X \) consider the following sequence:

\[
X \overset{s_X}{\longrightarrow} \text{mpcc}(X) \overset{s_{\text{mpcc}}(X)}{\longrightarrow} \text{mpcc}^2(X) \overset{s_{\text{mpcc}^2}(X)}{\longrightarrow} \ldots
\]

Note that every map in this sequence is an isometric embedding. We denote the metric direct limit of this sequence by \( \text{mpcc}^+(X) \) and let \( \text{mpcc}^{++}(X) \) be the completion of \( \text{mpcc}^+(X) \). In the sequel, we identify the spaces \( \text{mpcc}^n(X) \) with the corresponding subspaces of \( \text{mpcc}^+(X) \) and \( \text{mpcc}^{++}(X) \).

Denote by \( \text{mpcc}^\omega(X) \) the inverse limit of the sequence

\[
\text{mpcc}(X) \leftarrow u_X \text{mpcc}^2(X) \leftarrow u_{\text{mpcc}^2}(X) \text{mpcc}^3(X) \leftarrow u_{\text{mpcc}^3}(X) \ldots
\]

Let \( \psi_n : \text{mpcc}^\omega(X) \rightarrow \text{mpcc}^n(X) \) denote the natural projection.
There exists a natural embedding $\theta: \text{mpcc}^+(X) \to \text{mpcc}^{\omega}(X)$. The restriction of this embedding onto the set $\text{mpcc}^n(X)$ is uniquely determined by the maps

$$s_{nm} = s_{\text{mpcc}^{m-1}(X)} \ldots s_{\text{mpcc}^n(X)}: \text{mpcc}^n(X) \to \text{mpcc}^m(X), \; n < m.$$  

We write $\theta = (\theta_n)$, where $\theta_n = \psi_n \theta$.

The following proposition is proved in [4] in general form; in turn, this is a generalization of a result from [14].

**Proposition 3.3.** The (unique) extension $\bar{\theta}: \text{mpcc}^{++}(X) \to \text{mpcc}^{\omega}(X)$ of the map $\theta$ is an embedding.

**Proof.** Similarly as in [10, Lemma 3], one can prove that the map $\bar{\theta}$ is injective. We are going to show that the map $\bar{\theta}^{-1}$ is continuous. To this end, for any $x \in \text{mpcc}^{++}(X)$ and $\varepsilon > 0$ one should find a neighborhood $U$ of $\bar{\theta}(x)$ in $\text{mpcc}^\omega(X)$ such that

$$\bar{\theta}^{-1}(U) \subset B_\varepsilon(x). \quad (3.1)$$

We write $\bar{\theta} = (\bar{\theta}_i)$, where $\bar{\theta}_i = \psi_i \circ \bar{\theta}$. Again, similarly as in [10, Lemma 3], the sequence $(\bar{\theta}_i(x))$ converges to $x$ and therefore there exists $n$ such that

$$d(\bar{\theta}_k(x), x) < \varepsilon/4 \quad \text{for all} \; k \geq n. \quad (3.2)$$

Put

$$V = O_{\varepsilon/4}(\bar{\theta}_n(x))) \subset \text{mpcc}^{n+1}(X), \quad U = \psi^{-1}_{n+1}(V).$$

Let us verify the inclusion

$$\text{mpcc}^+(X) \cap \bar{\theta}^{-1}(U) \subset O_{3\varepsilon/4}(x). \quad (3.3)$$

Let $y \in \text{mpcc}^+(X) \cap \bar{\theta}^{-1}(U)$. Then there exists $k \geq n + 1$ such that $y \in \text{mpcc}^k(X) \subset \text{mpcc}^+(X)$. Since $y \in \bar{\theta}^{-1}(U)$, we have $\psi_{n+1}(y) \in V$. Since

$$\varepsilon/2 > d(\psi_{n+1}(y), \bar{\theta}_n(x)),$$

from (3.2) and Lemma 3.1 it follows that

$$d(y, x) \leq d(y, \bar{\theta}_n(x)) + d(\bar{\theta}_n(x), x) < \varepsilon/2 + \varepsilon/4 + 3\varepsilon/4.$$

Therefore, the inclusion in (3.3) is verified. Since $\text{mpcc}^+(X)$ is dense in $\text{mpcc}^{++}(X)$, from (3.3) we obtain that $\bar{\theta}^{-1}(U) \subset B_{3\varepsilon/4}(x) \subset O_\varepsilon(x)$. \hfill $\square$

Let $Q = [-1, 1]^\omega$ be the Hilbert cube, $s = (-1, 1)^\omega$ be its pseudointerior and $\text{rint} \; Q = \{(x_i) \in Q \mid \sup_i |x_i| < 1\}$ be its radial interior.

**Theorem 3.4.** Let $X$ be a compact max-plus convex body in $\mathbb{R}^n$. Then the triple $(\text{mpcc}^\omega(X), \text{mpcc}^{++}(X), \text{mpcc}^+(X))$ is homeomorphic to the triple $(Q, s, \text{rint} \; Q)$. 


Proof. Consider the following metric \( \rho \) on \( \text{mpcc}^\omega(X) \),
\[
\rho((x_i), (y_i)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i},
\]
where
\[
(x_i), (y_i) \in \text{mpcc}^\omega(X) \subset \prod_{i=1}^{\infty} \text{mpcc}^i(X).
\]

For every \( i \in \mathbb{N} \) and \( j > i \), the maps
\[
q_{ij} = \text{mpcc}(s_{i-1,j-1}): \text{mpcc}^i(X) \to \text{mpcc}^j(X)
\]
determine the map \( q_i: \text{mpcc}^i(X) \to \text{mpcc}^\omega(X) \).

We first show that the pair \((\text{mpcc}^\omega(X), \text{mpcc}^{++}(X))\) is homeomorphic to the pair \((Q, s)\). By [1, Theorems 2.3 and 3.3, Chapter V], it suffices to prove that the set
\[
B = \text{mpcc}^\omega(X) \setminus \text{mpcc}^{++}(X)
\]
is a \( Z \)-skeletoid in \( \text{mpcc}^\omega(X) \). Note first that the set \( B \) is \( \sigma \)-compact as the complement to the topologically complete set \( \text{mpcc}^{++}(X) \). Note also that every compact subset \( K \subset B \) is a \( Z \)-set in \( \text{mpcc}^\omega(X) \). Indeed, the sequence of retractions \( \psi_n: \text{mpcc}^\omega(X) \to \text{mpcc}^n(X) \) converges uniformly to the identity map of \( \text{mpcc}^\omega(X) \) and the image of every \( \psi_n \) misses \( K \). By [1, Theorems 3.2, Chapter V], in order to show that \( B \) is a \( Z \)-skeletoid it suffices to find a \( Z \)-skeletoid in \( B \). In turn, it suffices to find a sequence a sequence \((L_i)\) of compact subspaces in \( B \) such that:

1. every \( L_i \) is homeomorphic to \( Q \);
2. every \( L_i \) is a \( Z \)-set in \( Q_{i+1} \);
3. for every \( i \) there is a retraction \( r_i: \text{mpcc}^\omega(X) \to L_i \) and the sequence \((r_i)\) of retractions uniformly converges to the identity map.

The construction of such a sequence \((L_i)\) is analogous to that in the proof of [4, Theorem 4], therefore we drop the details. We suppose that \( X \) is a max-plus convex body in \( \mathbb{R}^n \), \( n \geq 2 \). Also, we suppose that \( \text{diam} X \leq 1 \). Then \( \text{diam} \text{mpcc}^\omega(X) \leq 1 \).

By \( K_1 \) we denote the set
\[
\{ A \in \text{mpcc}(X) | \text{ there is } x \in A \text{ such that } x + (\varepsilon, \ldots, \varepsilon) \in A \},
\]
where \( \varepsilon > 0 \). Clearly, \( K_1 \) is max-plus convex and if \( \varepsilon \) is small enough then \( K_1 \) is nonempty and can be made as close to \( \text{mpcc}(X) \) as we wish. We require that there is a retraction \( r_1 \) of \( \text{mpcc}(X) \) onto \( K_1 \) which is 1-close to the identity. Let \( L_1 = q_1(K_1) \).

Assuming that \( K_i, i \leq p \), are already constructed we let
\[
K_{p+1} = \{ A \in \text{mpcc}^p(X) | \text{ there is } x \in A \text{ such that } x + (\varepsilon, \ldots, \varepsilon) \in A \},
\]
where $\varepsilon > 0$ is chosen small enough that

$$\text{mpcc}(s_{\text{mpcc}^{-1}}(X))(K_p) \subset K_{p+1}$$

and there is a retraction

$$r_{p+1}: \text{mpcc}^{p+2}(X) \to K_{p+1}$$

which is $2^{-p}$-close to the identity. Let $L_{p+1} = q_{p+1}(K_{p+1})$.

Thus, $L = \bigcup_{i=1}^{\infty} L_i$ is a $Z$-skeletoid in $\text{mpcc}^\omega(X)$. We conclude that the pair $(\text{mpcc}^\omega(X), \text{mpcc}^{++}(X))$ is homeomorphic to $(Q,s)$.

Similarly, one can prove that $\text{mpcc}^+(X)$ is a $Z$-skeletoid in $\text{mpcc}^\omega(X)$. Therefore, the pair $(\text{mpcc}^\omega(X), \text{mpcc}^{++}(X))$ is homeomorphic to $(Q,\text{rint}Q)$.

We now apply [4, Theorem 2] to finish the proof. □

4. Remarks and open questions

It is plausible that the main result can be extended to the case of all max-plus convex subsets of $\mathbb{R}^\alpha$, $\alpha \leq \omega$, of dimension $\leq 1$.

We also conjecture that there is a counterpart of the main result for the hyperspace of max-min convex sets in $\mathbb{R}^\tau$. Given $\lambda \in \mathbb{R}_{\text{max}} \cup \{\infty\}$ and $x = (x_\alpha) \in \mathbb{R}^\tau$, we define $\lambda \otimes x = (\min\{\lambda, x_\alpha\})$. A subset $A$ in $\mathbb{R}^\tau$ is said to be max-min convex if $\alpha \otimes a \oplus b \in A$ for all $a,b \in A$ and $\alpha \in \mathbb{R}_{\text{max}}$.

References


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