A new curvature-like tensor in an almost contact Riemannian manifold

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Memories of Professor Victor Kuzakon at ONAFT

Abstract. In a M. Prvanović's paper [5], we can find a new curvature-like tensor in an almost Hermitian manifold.

In this paper, we define a new curvature-like tensor, named contact holomorphic Riemannian, briefly (CHR), curvature tensor in an almost contact Riemannian manifold. Then, using this tensor, we mainly research (CHR)-curvature tensor in a Kenmotsu and a Sasakian manifold. We introduce the flatness of a (CHR)-curvature tensor and show that a Kenmotsu and a Sasakian manifold with a flat (CHR)-curvature tensor is flat, see Theorems 3.1 and 4.1. Next, we introduce the notion of an (CHR)- η -Einstein in an almost contact Riemannian manifold. In particular, in a Sasakian or a Kenmotsu manifold, a (CHR)- η -Einstein manifold is η -Einstein, see Theorem 5.3. Finally, from this tensor, we introduce a notion of a (CHR)-space form in an almost contact Riemannian manifold. In particular, if a Kenmotsu and a Sasakian manifold are (CHR)-space form, then the (CHR)-curvature tensor satisfies a special equation, see Theorems 6.2 and 7.1.

1. ALMOST CONTACT RIEMANNIAN MANIFOLDS

A real $(2n+1)$ -dimensional differentiable Riemannian manifold (M, g) is said to be an almost contact Riemannian manifold if it has a $(1, 1)$ -tensor φ and a 1-form η which satisfy [6]

$$\begin{aligned}\varphi^2 &= -I + \eta \otimes \xi, & \eta(\varphi X) &= 0, & \eta(\xi) &= 1, \\ g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y),\end{aligned}\tag{1.1}$$

for any $Y, X \in TM$, where ξ is defined by $g(\xi, X) = \eta(X)$, and TM is the tangent bundle of M . From $(1.1)_2$ the vector field ξ is a unit vector field

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and we call it the *structure vector field* of the almost contact Riemannian manifold. Next, in an almost contact Riemannian manifold M , we define a 2-form F as $F(X, Y) = g(\varphi X, Y)$ for any $X, Y \in TM$. Then the 2-form F is skew-symmetric and we call it the *fundamental 2-form* of this almost contact Riemannian manifold.

In an almost contact Riemannian manifold a 2-dimensional distribution spanned by a unit vector field X and φX is called a φ -*section* of X . The sectional curvature $R(X, \varphi X, \varphi X, X)$ is said to be the φ -*holomorphic sectional curvature* of X , where R denotes the Riemannian curvature tensor with respect to g .

An almost contact Riemannian manifold (M, φ, g, ξ) is called a *Kenmotsu manifold*, [1], [2], [3], if the structure tensors satisfy

$$\begin{aligned}(\nabla_X \varphi)Y &= -\eta(Y)\varphi X - g(X, \varphi Y)\xi, \\ \nabla_X \xi &= X - \eta(X)\xi,\end{aligned}$$

for any $X, Y \in TM$, where ∇ means the covariant derivation with respect to g . In a Kenmotsu manifold (M, φ, g, ξ) we know

$$\begin{aligned}(\nabla_X \eta)Y &= g(\varphi X, \varphi Y), \\ \eta(R(X, Y)Z) &= \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \\ R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\ R(\xi, X)Y &= \eta(Y)X - g(X, Y)\xi, \\ R_1(\varphi X, \varphi Y) &= R_1(X, Y) + 2n\eta(X)\eta(Y), \\ R_1(X, \xi) &= -2n\eta(X),\end{aligned} \tag{1.2}$$

for any $X, Y, Z, W \in TM$, where R_1 denotes the Ricci tensor with respect to g .

Moreover, in a Kenmotsu manifold we get

$$\begin{aligned}R(X, Y, \varphi Z, \varphi W) &= R(X, Y, Z, W) + \\ &\quad + g(\varphi X, Z)g(\varphi Y, W) - g(\varphi X, W)g(\varphi Y, Z) + \\ &\quad + g(X, W)g(Y, Z) - g(X, Z)g(Y, W), \\ R(X, Y, \varphi Z, W) &= R(X, Y, Z, \varphi W) + \\ &\quad + g(\varphi X, W)g(Y, Z) - g(\varphi Y, W)g(X, Z) + \\ &\quad + g(\varphi Y, Z)g(X, W) - g(\varphi X, Z)g(Y, W), \\ R(X, Y, \varphi Z, \varphi W) &= R(\varphi X, \varphi Y, Z, W), \\ R(\varphi X, \varphi Y, \varphi Z, \varphi W) &= R(X, Y, Z, W) + \\ &\quad + \eta(X)\eta(W)g(Y, Z) - \eta(X)\eta(Z)g(Y, W) + \\ &\quad + \eta(Y)\eta(Z)g(X, W) - \eta(Y)\eta(W)g(X, Z).\end{aligned} \tag{1.3}$$

A Kenmotsu manifold with constant φ -holomorphic sectional curvature c is called a *Kenmotsu space form* with c . Then, [1], its curvature tensor R is expressed by

$$\begin{aligned} R(X, Y, Z, W) = & \frac{c-3}{4} \left\{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \right\} + \\ & + \frac{c+1}{4} \left[\left\{ \eta(X)g(Y, W) - \eta(Y)g(X, W) \right\} \eta(Z) + \right. \\ & + \left\{ \eta(Y)g(X, Z) - \eta(X)g(Y, Z) \right\} \eta(W) + \\ & + F(Y, Z)F(X, W) - F(X, Z)F(Y, W) - \\ & \left. - 2F(X, Y)F(Z, W) \right]. \end{aligned} \quad (1.4)$$

A Kenmotsu manifold is said to be *flat* if its φ -sectional curvature is equal to zero on M .

An almost contact Riemannian manifold (M, φ, g, ξ) is called a *normal contact Riemannian* or *Sasakian* manifold if the structure vector field ξ and the fundamental 2-form F satisfy

$$\nabla_X \xi = \varphi X, \quad (1.5)$$

$$(\nabla_X F)(Y, Z) = \eta(Y)g(X, Z) - \eta(Z)g(X, Y) \quad (1.6)$$

for any $X, Y, Z \in TM$.

In a Sasakian manifold the Riemannian curvature tensor R with respect to g satisfies

$$\begin{aligned} R(X, Y, Z, \xi) &= \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \\ R(X, Y, \varphi Z, W) - R(X, Y, Z, \varphi W) &= \\ &= g(\varphi X, Z)g(Y, W) - g(\varphi X, W)g(Y, Z) + \\ &+ g(\varphi Y, W)g(X, Z) - g(\varphi Y, Z)g(X, W), \\ R(X, Y, \varphi Z, \varphi W) &= R(\varphi X, \varphi Y, Z, W) = R(X, Y, Z, W) + \\ &+ g(X, Z)g(Y, W) - g(Y, Z)g(X, W) + \\ &+ F(X, W)F(Y, Z) - F(Y, W)F(X, Z), \\ R(\varphi X, \varphi Y, \varphi Z, \varphi W) &= R(X, Y, Z, W) + \\ &+ \eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(W)g(Y, Z) + \\ &+ \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W), \\ R_1(X, \xi) &= 2n\eta(X) \end{aligned} \quad (1.7)$$

for any $X, Y, Z, W \in TM$.

A Sasakian manifold is said to be a *Sasakian space form* if it has constant φ -holomorphic sectional curvature. Then the curvature tensor field R

satisfies ([4])

$$\begin{aligned}
 R(X, Y, Z, W) = & \frac{c+3}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\} + \\
 & + \frac{c-1}{4} \left[\eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + \right. \\
 & + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W) + \\
 & + F(Y, Z)F(X, W) - F(X, Z)F(Y, W) - \\
 & \left. - 2F(X, Y)F(Z, W) \right], \tag{1.8}
 \end{aligned}$$

where c is a constant holomorphic sectional curvature.

A Sasakian space form with zero holomorphic sectional curvature is called *flat*.

An almost contact Riemannian manifold M^{2n+1} is said to be η -Einstein if the Ricci tensor R_1 with respect to g satisfies

$$R_1(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for certain differentiable functions a and b on M^{2n+1} which are called the *associated functions* of R_1 . In particular, in an η -Einstein Kenmotsu and an η -Einstein Sasakian manifold, associated functions a and b respectively satisfy the following relations

$$a + b = -2n, \quad \tau = 2n(a - 1), \tag{1.9}$$

and

$$a + b = 2n, \quad \tau = 2n(a + 1), \tag{1.10}$$

where τ is the scalar curvature with respect to g .

2. A NEW CURVATURE-LIKE TENSOR FIELD IN AN ALMOST CONTACT RIEMANNIAN MANIFOLD

In this section, we define a new curvature-like tensor field in an almost contact Riemannian manifold.

In a differentiable manifold M a $(0, 4)$ -type tensor field $T(X, Y, Z, W)$ is called *curvature-like* if it satisfies

$$\begin{aligned}
 T(X, Y, Z, W) = & -T(Y, X, Z, W) = -T(X, Y, W, Z) = T(Z, W, X, Y), \\
 T(X, Y, Z, W) + & T(X, Z, W, Y) + T(X, W, Y, Z) = 0,
 \end{aligned}$$

for any $X, Y, Z, W \in TM$. Of course, a Riemannian curvature tensor, a conformal curvature tensor in a Riemannian manifold, a Bochner curvature tensor in a Kählerian manifold and a C -Bochner curvature tensor in a Sasakian manifold are examples of curvature-like tensor fields.

In an almost contact Riemannian manifold (M, φ, ξ, g) , we define a $(0, 4)$ -type tensor field $(CHR)(X, Y, Z, W)$ as

$$\begin{aligned}
 (CHR)(X, Y, Z, W) = & \frac{1}{16} \left\{ 3[R(X, Y, Z, W) + R(\varphi X, \varphi Y, Z, W) + \right. \\
 & + R(X, Y, \varphi Z, \varphi W) + R(\varphi X, \varphi Y, \varphi Z, \varphi W)] - \\
 & - R(X, Z, \varphi W, \varphi Y) - R(\varphi X, \varphi Z, W, Y) - \\
 & - R(X, W, \varphi Y, \varphi Z) - R(\varphi X, \varphi W, Y, Z) + \\
 & + R(\varphi X, Z, \varphi W, Y) + R(X, \varphi Z, W, \varphi Y) + \\
 & + R(\varphi X, W, Y, \varphi Z) + R(X, \varphi W, \varphi Y, Z) + \\
 & + \eta(X)[2\{R(Z, W, Y, \xi) + R(\varphi Z, \varphi W, Y, \xi)\} + R(\varphi Z, \varphi Y, W, \xi) + \\
 & + R(\varphi Y, \varphi W, Z, \xi) - R(\varphi Y, W, \varphi Z, \xi) - R(Z, \varphi Y, \varphi W, \xi)] - \\
 & - \eta(Y)[2\{R(Z, W, X, \xi) + R(\varphi Z, \varphi W, X, \xi)\} + R(\varphi Z, \varphi X, W, \xi) + \\
 & + R(\varphi X, \varphi W, Z, \xi) - R(Z, \varphi X, \varphi W, \xi) - R(\varphi X, W, \varphi Z, \xi)] + \\
 & + \eta(Z)[2\{R(X, Y, W, \xi) + R(\varphi X, \varphi Y, W, \xi)\} + R(\varphi X, \varphi W, Y, \xi) + \\
 & + R(\varphi W, \varphi Y, X, \xi) - R(\varphi W, Y, \varphi X, \xi) - R(X, \varphi W, \varphi Y, \xi)] - \\
 & - \eta(W)[2\{R(X, Y, Z, \xi) + R(\varphi X, \varphi Y, Z, \xi)\} + R(\varphi X, \varphi Z, Y, \xi) + \\
 & + R(\varphi Z, \varphi Y, X, \xi) - R(X, \varphi Z, \varphi Y, \xi) - R(\varphi Z, Y, \varphi X, \xi)] \left. \right\}
 \end{aligned}$$

for any $X, Y, Z, W \in TM$. Then, by the straightforward calculation, we can check the above tensor field is curvature-like. We call this tensor field a *contact holomorphic Riemannian curvature tensor* in an almost contact Riemannian manifold.

A contact Riemannian manifold is said to be (CHR) -flat if the (CHR) -curvature tensor vanishes, identically.

In an almost contact Riemannian manifold the (CHR) -curvature tensor satisfies

$$\begin{aligned}
 16(CHR)(\varphi X, \varphi Y, Z, W) = & 16(CHR)(X, Y, Z, W) + \\
 & + \eta(X) \left\{ R(Z, W, Y, \xi) + R(\varphi Z, \varphi W, Y, \xi) \right\} - \\
 & - \eta(Y) \left\{ R(Z, W, X, \xi) + R(\varphi Z, \varphi W, X, \xi) \right\} + \\
 & + T(X, Y, Z, W), \\
 16(CHR)(X, Y, \varphi Z, \varphi W) = & 16(CHR)(X, Y, Z, W) + \\
 & + \eta(Z) \left\{ R(X, Y, W, \xi) + R(\varphi X, \varphi Y, W, \xi) \right\} - \\
 & - \eta(W) \left\{ R(X, Y, Z, \xi) + R(\varphi X, \varphi Y, Z, \xi) \right\} + \\
 & + T(X, Y, Z, W),
 \end{aligned}$$

$$\begin{aligned}
16(CHR)(\varphi X, \varphi Y, \varphi Z, \varphi W) &= 16(CHR)(X, Y, Z, W) + \\
&+ \eta(X) \left\{ R(Z, W, Y, \xi) + R(\varphi Z, \varphi W, Y, \xi) \right\} - \\
&- \eta(Y) \left\{ R(Z, W, X, \xi) + R(\varphi Z, \varphi W, X, \xi) \right\} + \\
&+ \eta(Z) \left\{ R(X, Y, W, \xi) + R(\varphi X, \varphi Y, W, \xi) \right\} - \\
&- \eta(W) \left\{ R(X, Y, Z, \xi) + R(\varphi X, \varphi Y, Z, \xi) \right\} - \\
&- \eta(Z) \left\{ \eta(X) R(\xi, Y, W, \xi) + \eta(Y) R(X, \xi, W, \xi) \right\} + \\
&+ \eta(W) \left\{ \eta(X) R(\xi, Y, Z, \xi) + \eta(Y) R(X, \xi, Z, \xi) \right\} + \\
&+ T(X, Y, Z, W), \\
(CHR)(X, Y, Z, \xi) &= \frac{1}{16} \left[R(X, Y, Z, \xi) + R(\varphi X, \varphi Y, Z, \xi) + \right. \\
&+ \eta(X) \left\{ 2R(Y, \xi, Z, \xi) - R(\varphi Y, \xi, \varphi Z, \xi) \right\} - \\
&\left. - \eta(Y) \left\{ 2R(X, \xi, Z, \xi) - R(\varphi X, \xi, \varphi Z, \xi) \right\} \right],
\end{aligned}$$

where we put

$$\begin{aligned}
T(X, Y, Z, W) &= \eta(Z) \left\{ -2\eta(X) R(\xi, Y, W, \xi) - 2\eta(Y) R(X, \xi, W, \xi) + \right. \\
&+ \eta(X) R(\xi \varphi W, \varphi Y, \xi) + \eta(Y) R(\varphi W, \xi, \varphi X, \xi) \left. \right\} - \\
&- \eta(W) \left\{ -2\eta(X) R(\xi, Y, Z, \xi) - 2\eta(Y) R(X, \xi, Z, \xi) + \right. \\
&+ \eta(X) R(\xi, \varphi Z, \varphi Y, \xi) + \eta(Y) R(\varphi Z, \xi, \varphi X, \xi) \left. \right\}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(CHR)(X, \varphi X, \varphi X, X) &= \\
&= \frac{1}{16} \left[16R(X, \varphi X, \varphi X, X) + 8\eta(X) R(X, \varphi X, \varphi X, \xi) + \right. \\
&\quad \left. + \eta(X)^2 \{ 11R(\xi, \varphi X, \varphi X, \xi) + 4R(\xi, X, X, \xi) \} \right]. \tag{2.1}
\end{aligned}$$

For a unit vector field X the following tensor $(CHR)(X, \varphi X, \varphi X, X)$ is called a ϕ -holomorphic (CHR) -curvature of X , and an almost contact Riemannian manifold is called a (CHR) -space form if the holomorphic (CHR) -curvature constant on M .

3. (CHR) -CURVATURE TENSOR IN A KENMOTSU MANIFOLD

In this section we consider a (CHR) -curvature tensor in a Kenmotsu manifold. By virtue of (1.2) and (1.3) we have

$$\begin{aligned}
 (CHR)(X, Y, Z, W) &= R(X, Y, Z, W) + \\
 &+ \frac{1}{4} \left\{ g(\varphi X, Z)g(\varphi Y, W) - g(\varphi X, W)g(\varphi Y, Z) + 2g(\varphi X, Y)g(\varphi Z, W) + \right. \\
 &\quad + \eta(X)\eta(W)g(Y, Z) - \eta(X)\eta(Z)g(Y, W) + \\
 &\quad \left. + \eta(Y)\eta(Z)g(X, W) - \eta(Y)\eta(W)g(X, Z) \right\} + \\
 &+ \frac{3}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\}.
 \end{aligned} \tag{3.1}$$

Let the manifold be (CHR) -flat. Then we have from (3.1) that the curvature tensor R is written by

$$\begin{aligned}
 R(X, Y, Z, W) &= \frac{3}{4} \left\{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \right\} + \\
 &+ \frac{1}{4} \left\{ g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) - 2g(\varphi X, Y)g(\varphi Z, W) + \right. \\
 &\quad + \eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(W)g(Y, Z) + \\
 &\quad \left. + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W) \right\}.
 \end{aligned} \tag{3.2}$$

Comparing (3.2) and (1.4) we obtain

Theorem 3.1. *A (CHR) -flat Kenmotum manifold is flat.*

4. (CHR) -CURVATURE TENSOR IN A SASAKIAN MANIFOLD

In this section, we consider a (CHR) -flat Sasakian manifold and prove that this manifold is a Sasakian space form with zero holomorphic sectional curvature.

By virtue of (1.5), (1.6), (1.7) and Bianchi identity, [6], the (CHR) -curvature tensor (CHR) in a Sasakian manifold satisfies

$$\begin{aligned}
 (CHR)(X, Y, Z, W) &= R(X, Y, Z, W) + \\
 &+ \frac{3}{4} \left\{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \right\} + \\
 &+ \frac{1}{4} \left\{ g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) - 2g(\varphi X, Y)g(\varphi Z, W) + \right. \\
 &\quad + \eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(W)g(Y, Z) + \\
 &\quad \left. + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W) \right\}.
 \end{aligned} \tag{4.1}$$

Consider the (CHR) -flat case in a Sasakian manifold. Then we have from (3.1) that the curvature tensor R is written as

$$\begin{aligned} R(X, Y, Z, W) = & \frac{3}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\} - \\ & - \frac{1}{4} \left\{ g(\varphi X, W)g(\varphi Y, Z) - g(\varphi X, Z)g(\varphi Y, W) - 2g(\varphi X, Y)g(\varphi Z, W) + \right. \\ & + \eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(W)g(Y, Z) + \\ & \left. + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W) \right\}. \end{aligned}$$

Comparing the above equation and (1.3) we get

Theorem 4.1. *A (CHR) -flat Sasakian manifold is flat.*

5. THE PROPERTIES OF THE (CHR) -CURVATURE TENSOR

In this section we consider other properties of the (CHR) -curvature tensor in Kenmotsu and Sasakian manifolds.

Since the (CHR) -curvature tensor in a Kenmotsu manifold satisfies (3.1), using (1.3) and (2.1), we obtain

Proposition 5.1. *In a Kenmotsu manifold we have*

$$\begin{aligned} (CHR)(X, Y, Z, \xi) &= 0, \\ (CHR)(X, Y, \varphi Z, \varphi W) &= (CHR)(\varphi X, \varphi Y, Z, W) = \\ &= (CHR)(\varphi X, \varphi Y, \varphi Z, \varphi W) = (CHR)(X, Y, Z, W), \quad (5.1) \\ (CHR)(X, \varphi X, \varphi X, X) &= R(X, \varphi X, \varphi X, X) + \\ &+ \eta(X)^2 \{g(X, X) - \eta(X)^2\} \end{aligned}$$

Similarly, we have from (1.7) and (4.1)

Proposition 5.2. *In a Sasakian manifold, we have*

$$\begin{aligned} (CHR)(X, Y, Z, \xi) &= 0, \\ (CHR)(X, Y, \varphi Z, \varphi W) &= (CHR)(\varphi X, \varphi Y, Z, W) = \\ &= (CHR)(\varphi X, \varphi Y, \varphi Z, \varphi W) = (CHR)(X, Y, Z, W) \quad (5.2) \\ (CHR)(X, \varphi X, \varphi X, X) &= R(X, \varphi X, \varphi X, X) - \\ &- \eta(X)^2 \{g(X, X) - \eta(X)^2\}. \end{aligned}$$

Next, we put

$$\rho(CHR)(X, Y) = \sum_{i=1}^{2n+1} (CHR)(e_i, X, Y, e_i)$$

for a local orthonormal frame $(e_1, e_2, \dots, e_{2n+1})$ of an almost contact Riemannian manifold M^{2n+1} . We say this $\rho(CHR)(X, Y)$ a (CHR) -Ricci tensor. By virtue of (3.1) and (4.1), the (CHR) -Ricci tensor in a Kenmotsu and a Sasakian manifold are respectively written by

$$\begin{aligned} \rho(CHR)(X, Y) &= R_1(X, Y) + \\ &+ \frac{1}{2} \left\{ (3n-1)g(X, Y) + (n+1)\eta(X)\eta(Y) \right\}, \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \rho(CHR)(X, Y) &= R_1(X, Y) - \\ &- \frac{1}{2} \left\{ (3n-1)g(X, Y) + (n+1)\eta(X)\eta(Y) \right\}. \end{aligned} \quad (5.4)$$

We also put

$$\tau(CHR) = \sum_{i=1}^{2n+1} (CHR)(e_i, e_i)$$

which is called the (CHR) -scalar curvature of (CHR) -curvature tensor. Then, in a Kenmotsu and a Sasakian manifold, we respectively have from (5.3) and (5.4)

$$\begin{aligned} \tau(CHR) &= \tau + n(3n+1), \\ \tau(CHR) &= \tau - n(3n+1). \end{aligned}$$

Now, a contact Riemannian manifold M^{2n+1} is called (CHR) - η -Einstein if its (CHR) -Ricci tensor $\rho(CHR)(X, Y)$ has the form

$$\rho(CHR)(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y) \quad (5.5)$$

for certain functions α and β which are called *associated functions* of $\rho(CHR)$. In particular, if our manifold is Kenmotsu or Sasakian, then we have from (5.3) and (5.4) the Ricci tensors are respectively written as

$$R_1(X, Y) = \left(\alpha - \frac{3n-1}{2} \right) g(X, Y) + \left(\beta - \frac{n+1}{2} \right) \eta(X)\eta(Y), \quad (5.6)$$

$$R_1(X, Y) = \left(\alpha + \frac{3n-1}{2} \right) g(X, Y) + \left(\beta + \frac{n+1}{2} \right) \eta(X)\eta(Y). \quad (5.7)$$

Conversely, if our manifold M^{2n+1} is η -Einstein, then we can easily that a Kenmotsu and a Sasakian manifolds are (CHR) - η -Einstein. Moreover, we have from (1.9), (5.5) and (5.6)

$$\alpha + \beta = 4n \quad : \text{Kenmotsu}, \quad (5.8)$$

$$\alpha + \beta = 0 \quad : \text{Sasakian}. \quad (5.9)$$

By virtue of (5.3), (5.4), (5.8), and (5.9) the scalar curvature τ and the (CHR) -scalar curvature $\tau(CHR)$ are respectively written as

$$\tau = 2n\alpha - 3n(n-1), \quad \tau(CHR) = 2n(\alpha + 2) \quad : \text{Kenmotsu}, \quad (5.10)$$

$$\tau = 2n(\alpha + 3n + 1), \quad \tau(CHR) = n(2\alpha + 3n + 1) \quad : \text{Sasakian.} \quad (5.11)$$

Thus we have

Theorem 5.3. *A Kenmotsu or a Sasakian manifold M^{2n+1} is (CHR) - η -Einstein if and only if M^{2n+1} is η -Einstein and the scalar curvatures τ and $\tau(CHR)$ are respectively written by (5.10) and (5.11) which includes only the associated function α .*

Corollary 5.4. *In an (CHR) - η -Einstein Kenmotsu or Sasakian manifold M^{2n+1} , the scalar curvatures τ and $\tau(CHR)$ are constant if and only if one of the associated functions is constant.*

6. (CHR) -SPACE FORM

In this section we define a (CHR) -space form in an almost contact Riemannian manifold.

Let M^{2n+1} be an $(2n+1)$ -dimensional almost contact Riemannian manifold.

Definition 6.1. An almost contact Riemannian manifold M^{2n+1} is said to be a (CHR) -space form if its contact holomorphic (CHR) -sectional curvature is constant for any vector fields and any point on M^{2n+1}

$$(CHR)(X, \varphi X, \varphi X, X) = c\|X\|^2\{\|X\|^2 - \eta(X)^2\} \quad (6.1)$$

for a certain constant c and any $X \in TM$, where $\|X\|$ is the length of X with respect to g .

$$(CHR)(X, \varphi X, \varphi X, X) = c\|X\|^4$$

for any $X \in TM - \{\xi\}$.

First, we consider a Kenmotsu (CHR) -space form, that is a manifold M whose (CHR) -curvature tensor satisfies (6.1).

By virtue of (5.1)

$$c\|X\|^4 = R(X, \varphi X, \varphi X, X) + \eta(X)^2\{g(X, X) - \eta(X)^2\}$$

for any $X \in TM$. Moreover, from the above equation we know

$$c\|X\|^4 = R(X, \varphi X, \varphi X, X)$$

for any $X \in TM - \{\xi\}$. This means that the manifold is a Kenmotsu space form. So, using (3.1) and Proposition 5.1, the curvature tensor R is written

as

$$\begin{aligned}
 (CHR)(X, Y, Z, W) = & \frac{1}{4} \left\{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + \right. \\
 & + g(\varphi X, Z)g(\varphi Y, W) - g(\varphi X, W)g(\varphi Y, Z) + 2g(\varphi X, Y)g(\varphi Z, \varphi W) + \\
 & + \eta(X)\eta(W)g(Y, Z) - \eta(X)\eta(Z)g(Y, W) + \eta(Y)\eta(Z)g(X, W) - \\
 & \left. - \eta(Y)\eta(W)g(X, Z) \right\}. \tag{6.2}
 \end{aligned}$$

From the above equation, we obtain

$$\rho(CHR)(X, Y) = -\frac{n+1}{2} \left\{ g(X, Y) - \eta(X)\eta(Y) \right\}. \tag{6.3}$$

and

$$\tau(CHR) = -n(n+1). \tag{6.4}$$

Summing up, we have

Theorem 6.2. *If an $(2n+1)$ -dimensional (CHR) -Kenmotsu space form, then the (CHR) -curvature tensor, the (CHR) -Ricci tensor and the (CHR) -scalar curvature are respectively given by (6.2), (6.3), and (6.4).*

7. SASAKIAN (CHR) -SPACE FORMS

In this section, we consider a Sasakian (CHR) -space form and we determine the Riemannian curvature tensor of this manifold.

Let M be a Sasakian manifold. Then the (CHR) -curvature tensor satisfies (4.1). From this we have

$$\begin{aligned}
 (CHR)(X, \varphi X, \varphi X, X) = & R(X, \varphi X, \varphi X, X) - \\
 & - \eta(X)^2 g(X, X) + \eta(X)^4. \tag{7.1}
 \end{aligned}$$

Now suppose that M is a Sasakian (CHR) -space form. Then, we have from (6.1) and (7.1) that

$$R(X, \varphi X, \varphi X, X) = c\|X\|^4 + \eta(X)^2 g(X, X) - \eta(X)^4$$

for any $X \in TM$. In particular, for $X \in TM - \{\xi\}$ the above equation means

$$R(X, \varphi X, \varphi X, X) = c\|X\|^4,$$

that is, the manifold is a Sasakian space form with the constant φ -holomorphic sectional curvature c . So, its curvature tensor $R(X, Y, Z, W)$ satisfies (1.8). Substituting (1.8) into (4.1), we get

$$\begin{aligned} (CHR)(X, Y, Z, W) = & \frac{c}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) + \right. \\ & + \eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) + \eta(Y)\eta(W)g(X, Z) - \\ & - \eta(X)\eta(W)g(Y, Z) - g(\varphi X, W)g(\varphi Y, Z) + \\ & \left. + g(\varphi X, Z)g(\varphi Y, W) - 2g(\varphi X, Y)g(\varphi Z, W) \right\}. \end{aligned} \quad (7.2)$$

From (7.2), we obtain

$$\rho(CHR)(X, Y) = \frac{(n+1)c}{2} \left\{ g(X, Y) - \eta(X)\eta(Y) \right\} \quad (7.3)$$

and

$$\tau(CHR) = n(n+1)c. \quad (7.4)$$

As a result, we have

Theorem 7.1. *If M is an $(2n+1)$ -dimensional Sasakian (CHR) -space form, then its (CHR) -curvature tensor, the Ricci (CHR) -tensor and the scalar (CHR) -curvature are respectively given by (7.2), (7.3), and (7.4).*

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