Warped product semi-slant submanifolds in locally conformal Kaehler manifolds

Koji Matsumoto

Abstract. In 1994, in [13], N. Papaghiuc introduced the notion of semi-slant submanifold in a Hermitian manifold which is a generalization of CR - and slant-submanifolds. In particular, he considered this submanifold in Kaehlerian manifolds, [13]. Then, in 2007, V. A. Khan and M. A. Khan considered this submanifold in a nearly Kaehler manifold and obtained interesting results, [11]. Recently, we considered semi-slant submanifolds in a locally conformal Kaehler manifold and gave a necessary and sufficient conditions for two distributions (holomorphic and slant) to be integrable. Moreover, we considered these submanifolds in a locally conformal Kaehler space form, [4]. In this paper, we define 2-kind warped product semi-slant submanifolds in a locally conformal Kaehler manifold and consider some properties of these submanifolds.

1. INTRODUCTION

A Hermitian manifold $\widetilde{M}$ with structure $(J, \widetilde{g})$ is called a *locally conformal Kaehler* (an *l.c.K.-*) *manifold* if each point $x \in \widetilde{M}$ has an open neighbourhood U with differentiable function $\rho : U \rightarrow \mathcal{R}$ such that $\widetilde{g}^* = e^{-2\rho}\widetilde{g}|_U$ is a Kaehlerian metric on U , that is, $\nabla^* J = 0$, where J is the almost complex structure, $\widetilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\widetilde{g}^*$ and $\mathcal{R}$ is a real number space, [14]. A typical example of an l.c.K.-manifold which is not Kaehlerian is Hopf manifold, [14].

Then we know the following statement, see [10]:

2010 Mathematics Subject Classification: 53C40

Keywords: Locally conformal Kaehler manifold, slant distribution, semi-slant submanifold, warped product semi-slant submanifold

Proposition 1.1. *A Hermitian manifold $\widetilde{M}(J, \widetilde{g})$ is l.c.K. if and only if there exists a global closed 1-form α which is called Lee form satisfying*

$$(\widetilde{\nabla}_V J)U = -\widetilde{g}(\alpha^\sharp, U)JV + \widetilde{g}(V, U)\beta^\sharp + \widetilde{g}(JV, U)\alpha^\sharp - \widetilde{g}(\beta^\sharp, U)V \quad (1.1)$$

for any $V, U \in T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the covariant differentiation with respect to $\widetilde{g}$, $\alpha^\sharp$ is the dual vector field of α , the 1 form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β and $T\widetilde{M}$ is the tangent bundle of $\widetilde{M}$.

An l.c.K.-manifold $\widetilde{M}(J, \widetilde{g}, \alpha)$ is called an l.c.K.-space form if it has a constant holomorphic sectional curvature. Then, [10], the Riemannian curvature tensor $\widetilde{R}$ with respect to $\widetilde{g}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by the following formulas:

$$\begin{aligned} 4\widetilde{R}(X, Y, Z, W) = & c\{\widetilde{g}(X, W)\widetilde{g}(Y, Z) - \widetilde{g}(X, Z)\widetilde{g}(Y, W) + \\ & + \widetilde{g}(JX, W)\widetilde{g}(JY, Z) - \widetilde{g}(JX, Z)\widetilde{g}(JY, W) - \\ & - 2\widetilde{g}(JX, Y)\widetilde{g}(JZ, W)\} + \\ & + 3\{P(X, W)\widetilde{g}(Y, Z) - P(X, Z)\widetilde{g}(Y, W) + \\ & + \widetilde{g}(X, W)P(Y, Z) - \widetilde{g}(X, Z)P(Y, W)\} - \\ & - \widetilde{P}(X, W)\widetilde{g}(JY, Z) + \widetilde{P}(X, Z)\widetilde{g}(JY, W) - \\ & - \widetilde{g}(JX, W)\widetilde{P}(Y, Z) + \widetilde{g}(JX, Z)\widetilde{P}(Y, W) + \\ & + 2\{\widetilde{P}(X, Y)\widetilde{g}(JZ, W) + \widetilde{g}(JX, Y)\widetilde{P}(Z, W)\} \end{aligned} \quad (1.2)$$

for any $X, Y, Z, W \in T\widetilde{M}$, where P and $\widetilde{P}$ are respectively defined by

$$P(X, Y) = -(\widetilde{\nabla}_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\|\alpha\|^2\widetilde{g}(X, Y), \quad (1.3)$$

and

$$\widetilde{P}(X, Y) = P(JX, Y) \quad (1.4)$$

for any $X, Y \in T\widetilde{M}$, where $\|\alpha\|$ is the length of the Lee form α .

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Then we put $M = M_1 \times M_2$ be the product manifold of M_1 and M_2 . For a positive differentiable function f on M_2 , we define a Riemannian metric tensor g on M as

$$g(U, V) = e^{f^2}g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V) \quad (1.5)$$

for any $U, V \in TM$, where π_1 (resp. π_2) denotes the projection operator of M to M_1 (resp. M_2) and π_{1*} (resp. π_{2*}) is the differential of π_1 (resp. π_2). Then the Riemannian manifold M is called a *warped product manifold* of M_1 and M_2 with the warping function f and we write it $M_1 \otimes_f M_2$, [12].

Let ∇ , ∇_1 and ∇_2 be the covariant differentiation with respect to g , g_1 and g_2 , respectively. Then, we have from (1.5)

$$\begin{aligned}\nabla_X Y &= \nabla_{1X} Y - f^2 e^{f^2} g_1(X, Y)(d_2 \log f)^*, \\ \nabla_X Z &= \nabla_Z X = f^2(Z \log f)X, \\ \nabla_Z W &= \nabla_{2Z} W\end{aligned}\tag{1.6}$$

for any $X, Y \in TM_1$ and $Z, W \in TM_2$, where $d_2 \log f$ means the differential of $\log f$ and $(d_2 \log f)^*$ is the dual vector field of $d_2 \log f$.

By virtue of (1.6), the curvature tensor R with respect to g is written as

$$\begin{aligned}R(X_1, X_2, X_3, X_4) &= e^{f^2} [R_1(X_1, X_2, X_3, X_4) - \\ &\quad - f^4 e^{f^2} \|\nabla_2 \log f\|^2 \{g_1(X_1, X_4)g_1(X_2, X_3) - \\ &\quad - g_1(X_1, X_3)g_1(X_2, X_4)\}], \\ R(X_1, Z_1, Z_2, X_2) &= -f^2 e^{f^2} \{(2 + f^2)(Z_1 \log f)(Z_2 \log f) + \\ &\quad + \nabla_{2Z_1} \nabla_{2Z_2} \log f\} g_1(X_1, X_2), \\ R(Z_1, Z_2, Z_3, Z_4) &= R_2(Z_1, Z_2, Z_3, Z_4), \\ Other &= 0,\end{aligned}\tag{1.7}$$

and the Ricci tensor ρ with respect to g is separated as

$$\begin{aligned}\rho(X_1, X_2) &= \rho_1(X_1, X_2) - \\ &\quad - f^2 e^{f^2} \{(2 + n_1 f^2) \|\nabla_2 \log f\|^2 + \delta_2 d_2\} g_1(X_1, X_2), \\ \rho(X_1, Z_1) &= 0, \\ \rho(Z_1, Z_2) &= \rho_2(Z_1, Z_2) - \\ &\quad - n_1 f^2 \{(2 + f^2)(\nabla_{2Z_1} \log f)(\nabla_{2Z_2} \log f) + \nabla_{2Z_1} \nabla_{2Z_2} \log f\},\end{aligned}\tag{1.8}$$

for any $X_1, X_2 \in TM_1$ and $Z_1, Z_2 \in TM_2$, where R_1 (resp. R_2) is the Riemannian curvature tensor with respect to g_1 (resp. g_2) and ρ_1 (resp. ρ_2) is the Ricci tensor with respect to g_1 (resp. g_2), d_2 (resp. δ_2) means the differential (resp. codifferential) with respect to g_2 , $\|\nabla_2 \log f\|$ is the length of $\nabla_2 \log f$ with respect to g_2 and $n_1 = \dim M_1$.

Finally, if we respectively put τ , τ_1 and τ_2 the scalar curvature with respect to g , g_1 and g_2 , then from (1.8), we can easily have

$$\tau = e^{f^2} \tau_1 + \tau_2 - (n_1 - 1)n_1 f^4 \|\nabla_2 \log f\|^2.\tag{1.9}$$

2. SEMI-SLANT-SUBMANIFOLDS IN AN ALMOST HERMITIAN MANIFOLD

In general, between a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and its Riemannian submanifold (M, g) , we know the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad \widetilde{\nabla}_X N = -A_N X + \nabla^\perp_X N \quad (2.1)$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where ∇ is the covariant differentiation with respect to g , σ is the second fundamental form and A_N is the shape operator with respect to N and $\nabla^\perp$ is the normal connection, [6]. The second fundamental form σ and the shape operator A are related by $\widetilde{g}(A_N Y, X) = \widetilde{g}(\sigma(Y, X), N)$ for any $Y, X \in TM$ and $N \in T^\perp M$.

The Gauss equation is given by

$$\begin{aligned} \widetilde{R}(U, V, W, Z) &= R(U, V, W, Z) + \widetilde{g}(\sigma(U, Z), \sigma(V, W)) \\ &\quad - \widetilde{g}(\sigma(U, W), \sigma(V, Z)), \end{aligned} \quad (2.2)$$

for any $U, V, W, Z \in TM$, [6].

A submanifold M is said to be totally geodesic, if the second fundamental form σ identically vanishes, [6].

We recall a warped product submanifold in a Riemannian manifold.

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold. A submanifold (M, g) is called a *warped product submanifold* of $\widetilde{M}$ if it satisfies

- (i) M is a product manifold of 2 submanifolds M_1 and M_2 ,
- (ii) two submanifolds are orthogonal with respect to $\widetilde{g}$,
- (iii) for certain Riemannian metric g_1 in M_1 , g_2 in M_2 and a certain positive differentiable function f in M_2 , the metric tensor g is defined by

$$g(U, V) = e^{f^2} g_1(\pi_{1*} U, \pi_{1*} V) + g_2(\pi_{2*} U, \pi_{2*} V) \quad (2.3)$$

for any $U, V \in TM$ is the induced metric of $\widetilde{g}$, [5].

By virtue of (1.7) and (2.3) the Riemannian curvature $\widetilde{R}$ is separated as

$$\begin{aligned} \widetilde{R}(X_1, X_2, X_3, X_4) &= e^{f^2} \{ R_1(X_1, X_2, X_3, X_4) \\ &\quad - f^4 e^{f^2} \|\log f\|^2 g_1(X_1, X_4) g_1(X_2, X_3) - g_1(X_1, X_3) g_1(X_2, X_4) \} \\ &\quad + \widetilde{g}(\sigma(X_1, X_4), \sigma(X_2, X_3)) - \widetilde{g}(\sigma(X_1, X_3), \sigma(X_2, X_4)), \\ \widetilde{R}(X_1, X_2, X_3, Z_1) &= \widetilde{g}(\sigma(X_1, Z_1), \sigma(X_2, X_3)) - \widetilde{g}(\sigma(X_1, X_3), \sigma(X_2, Z_1)), \\ \widetilde{R}(X_1, X_2, Z_1, Z_2) &= \widetilde{g}(\sigma(X_1, Z_2), \sigma(X_2, Z_1)) - \widetilde{g}(\sigma(X_1, Z_1), \sigma(X_2, Z_2)), \\ \widetilde{R}(X_1, Z_1, Z_2, X_2) &= -f^2 e^{f^2} \{ (2 + f^2)(Z_1 \log f)(Z_2 \log f) \\ &\quad + \nabla_{Z_2} \nabla_{Z_1} \log f \} g_1(X_1, X_2) + \widetilde{g}(\sigma(X_1, X_2), \sigma(Z_1, Z_2)) \end{aligned} \quad (2.4)$$

$$- \tilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, X_2)),$$

$$\tilde{R}(X_1, Z_1, Z_2, Z_3) = \tilde{g}(\sigma(X_1, Z_3), \sigma(Z_1, Z_2)) - \tilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, Z_3)),$$

$$\begin{aligned} \tilde{R}(Z_1, Z_2, Z_3, Z_4) = & R_2(Z_1, Z_2, Z_3, Z_4) + \\ & + \tilde{g}(\sigma(Z_1, Z_4), \sigma(Z_2, Z_3)) - \tilde{g}(\sigma(Z_1, Z_3), \sigma(Z_2, Z_4)), \end{aligned}$$

for any $X_1, X_2, X_3, X_4 \in TM_1$ and $Z_1, Z_2, Z_3, Z_4 \in TM_2$, where R_1 (resp. R_2) is the Riemannian curvature tensor with respect to g_1 (resp. g_2).

For a vector field $U \in TM$, the angle between JU and TM is called the *Wirtingar angle* of U .

A differentiable distribution $\mathcal{D}^\theta : x \rightarrow \mathcal{D}^\theta_x$ on M is said to be a *slant* one if for each $U_x \in \mathcal{D}^\theta_x$, the Wirtingar angle of U_x is constant ($= \theta$) for any $x \in M$. In this case, the Wirtingar angle is said to be the *slant angle*. In particular, if TM is slant, then the submanifold is called a *slant* one, [9]. A slant submanifold is *holomorphic* (resp. *totally real*) if its slant angle $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$). A slant submanifold is said to be *proper* if it is neither holomorphic nor totally real.

A submanifold M in $\widetilde{M}$ is called a *semi-slant submanifold* if there exists a differentiable distribution $\mathcal{D} : x \rightarrow \mathcal{D}_x \subset T_x M$ on M satisfying the following conditions:

- (i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = \mathcal{D}_x$ for each $x \in M$ and
- (ii) the complementary orthogonal distribution $\mathcal{D}^\theta : x \rightarrow \mathcal{D}^\theta_x \subset T_x M$ is slant with slant angle θ , where $T_x M$ is the tangent vector space of M at x , [13].

Remark 2.1. A semi-slant submanifold is a *CR*-submanifold if the slant angle is equal to $\frac{\pi}{2}$, [1], [2], [3], [7], [8], etc.

A semi-slant submanifold M is said to be *proper* if it is neither *CR*-, holomorphic, nor totally real.

In a submanifold M of an almost Hermitian manifold $\widetilde{M}(J, \tilde{g})$, for any $U \in TM$ and $\xi \in T^\perp M$, we write

$$JU = TU + FU, \quad J\xi = t\xi + h\xi, \quad (2.5)$$

where TU (resp. FU) means the tangential (resp. normal) component of JU and $t\xi$ (resp. $h\xi$) means the tangential (resp. normal) component of $J\xi$.

For a semi-slant submanifold M of an almost Hermitian manifold $\widetilde{M}$, the tangent bundle TM and the normal bundle $T^\perp M$ of M are decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^\theta, \quad T^\perp M = F\mathcal{D}^\theta \oplus \nu, \quad (2.6)$$

where ν denotes the orthogonal complementary distribution of $F\mathcal{D}^\theta$ in $T^\perp M$.

Further, in a semi-slant submanifold M we write

$$U = T_1U + T_2U, \quad (2.7)$$

for any $U \in TM$, where T_1U (resp. T_2U) denotes the $\mathcal{D}$ (resp. $\mathcal{D}^\theta$) component of U .

By virtue of (2.7) and (2.7), we can write

$$JU = JT_1U + TT_2U + FT_2U, \quad (2.8)$$

where $JT_1U \in \mathcal{D}$, $TT_2U \in \mathcal{D}^\theta$ and $FT_2U \in F\mathcal{D}^\theta \subset T^\perp M$. Thus if we put

$$QU = JT_1U + TT_2U \quad (2.9)$$

for any $U \in TM$, then Q is an automorphism on TM .

The covariant differentiation $\bar{\nabla}$ of T_1 , T_2 , T , F , t and h are defined as

$$\begin{aligned} (\bar{\nabla}_U T_1)V &= \nabla_U(T_1V) - T_1\nabla_U V, \\ (\bar{\nabla}_U T_2)V &= \nabla_U(T_2V) - T_2\nabla_U V, \\ (\bar{\nabla}_U T)V &= \nabla_U(TV) - T\nabla_U V, \\ (\bar{\nabla}_U F)V &= \nabla_U^\perp(FV) - F\nabla_U V, \\ (\bar{\nabla}_U t)\xi &= \nabla_U(t\xi) - t\nabla_U^\perp \xi, \\ (\bar{\nabla}_U h)\xi &= \nabla_U^\perp(h\xi) - h\nabla_U^\perp \xi \end{aligned} \quad (2.10)$$

for any $U, V \in TM$ and $\xi \in T^\perp M$.

Moreover, if we define the covariant differentiation $\bar{\nabla}$ of Q

$$(\bar{\nabla}_U Q)V = \nabla_U(QV) - Q\nabla_U V \quad (2.11)$$

for any $U, V \in TM$, then using (2.10), we have

$$\begin{aligned} (\bar{\nabla}_U Q)V &= (\tilde{\nabla}_U J)T_1V + J(\bar{\nabla}_U T_1)V + (\bar{\nabla}_U T)(T_2V) \\ &\quad + T(\bar{\nabla}_U T_2)V + J\sigma(U, T_1V) - \sigma(U, JT_1V) \end{aligned} \quad (2.12)$$

for any $U, V \in TM$. In particular, for any $X, Y \in \mathcal{D}$, the equation (2.12) is written as

$$(\bar{\nabla}_X Q)Y = (\tilde{\nabla}_X J)Y + FT_2\nabla_X Y + t\sigma(X, Y) + h\sigma(X, Y) - \sigma(X, TY). \quad (2.13)$$

Now, for $U, V \in TM$, we write

$$(\tilde{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V, \quad (2.14)$$

where $\mathcal{P}_U V$ (resp. $\mathcal{Q}_U V$) denotes the tangential (resp. normal) part of $(\tilde{\nabla}_U J)V$.

3. SEMI-SLANT SUBMANIFOLDS IN AN L.C.K.-MANIFOLD

Let M be a semi-slant submanifold of an l.c.K.-manifold $\widetilde{M}(J, \widetilde{g}, \alpha)$. Then we have from (1.1) and (2.14)

$$\begin{aligned} \mathcal{P}_U V &= -\widetilde{g}(\alpha_1^\sharp, V)TU + \widetilde{g}(U, V)(T\alpha_1^\sharp + t\alpha_2^\sharp) + \widetilde{g}(TU, V)\alpha_1^\sharp \\ &\quad - \widetilde{g}(T\alpha_1^\sharp + t\alpha_2^\sharp, V)U, \end{aligned} \quad (3.1)$$

$$\mathcal{Q}_U V = -\widetilde{g}(\alpha_1^\sharp, V)FU + \widetilde{g}(U, V)(F\alpha_1^\sharp + h\alpha_2^\sharp) + \widetilde{g}(TU, V)\alpha_2^\sharp,$$

where $\alpha_1^\sharp$ (resp. $\alpha_2^\sharp$) means the tangential (resp. normal) component of $\alpha^\sharp$.

In a semi-slant submanifold in an l.c.K.-manifold, we have from (3.1)₂

$$\mathcal{Q}_X Y - \mathcal{Q}_Y X = 2\widetilde{g}(TX, Y)\alpha_2^\sharp \quad (3.2)$$

for any $X, Y \in \mathcal{D}$.

Using theorems of V. A. Khan and M. A. Khan on integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ of a semi-slant submanifold in an almost Hermitian manifold, in [4], we proved

Proposition 3.1. (I) *The holomorphic distribution $\mathcal{D}$ of a semi-slant submanifold M in an l.c.K.-manifold $\widetilde{M}(J, \widetilde{g}, \alpha)$ is integrable if and only if*

$$\sigma(X, TY) - \sigma(Y, TX) = \mathcal{Q}_X Y - \mathcal{Q}_Y X = 2\widetilde{g}(TX, Y)\alpha_2^\sharp \quad (3.3)$$

for any $X, Y \in \mathcal{D}$.

(II) *The slant distribution $\mathcal{D}^\theta$ of a semi-slant submanifold M in an locally conformal Kaehler manifold $\widetilde{M}(J, \widetilde{g}, \alpha)$ is integrable if and only if*

$$\begin{aligned} T_1(\nabla_Z TW - \nabla_W TZ + A_{FZ}W - A_{FW}Z + \\ + \widetilde{g}(\alpha_1^\sharp, W)TZ - \widetilde{g}(\alpha_1^\sharp, Z)TW + 2\widetilde{g}(TW, Z)\alpha_1^\sharp) = 0 \end{aligned} \quad (3.4)$$

or equivalently

$$\begin{aligned} T_1\{(\bar{\nabla}_Z T)W - (\bar{\nabla}_W T)Z + T[Z, W] + A_{FZ}W - A_{FW}Z + \\ + \widetilde{g}(\alpha_1^\sharp, W)TZ - \widetilde{g}(\alpha_1^\sharp, Z)TW + 2\widetilde{g}(TW, Z)\alpha_1^\sharp\} = 0 \end{aligned} \quad (3.5)$$

for any $Z, W \in \mathcal{D}^\theta$.

4. WARPED PRODUCT SEMI-SLANT SUBMANIFOLDS IN L.C.K.-MANIFOLDS

Let $\mathcal{D}$ and $\mathcal{D}^\theta$ be two integrable distributions on a semi-slant submanifold M of an l.c.K.-manifold $\widetilde{M}(J, \widetilde{g}, \alpha)$. Then (3.3) and (3.4) hold true. Let also $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$) be the maximal integral submanifold of $\mathcal{D}$ (resp. $\mathcal{D}^\theta$). Then M is a product manifold of $M_{\mathcal{D}}$ and $M_{\mathcal{D}^\theta}$, that is,

$$M = M_{\mathcal{D}} \otimes M_{\mathcal{D}^\theta}. \quad (4.1)$$

We call the submanifold $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$) the *holomorphic* (resp. *slant*) *component* of M .

We define the following two type warped product submanifolds

$$M_1 := M_{\mathcal{D}} \otimes_{f_1} M_{\mathcal{D}^\theta} \quad (4.2)$$

for a certain differentiable function f_1 on $M_{\mathcal{D}^\theta}$ and

$$M_2 := M_{\mathcal{D}^\theta} \otimes_{f_2} M_{\mathcal{D}} \quad (4.3)$$

for a certain differentiable function f_2 on $M_{\mathcal{D}}$. We say that M_1 (resp. M_2) the *first* (resp. *second*) *type warped product semi-slant submanifold* of an l.c.K.-manifold.

In this paper, we mainly consider the first type warped product semi-slant submanifold.

Let M be the first type warped product semi-slant submanifold in an l.c.K.-manifold $\widetilde{M}$. Then the induced metric tensor g on M from $\widetilde{M}$ is given by

$$g(U, V) = e^{f_1^2} g_{\mathcal{D}}(\pi_{\mathcal{D}} * U, \pi_{\mathcal{D}} * V) + g_{\mathcal{D}^\theta}(\pi_{\mathcal{D}^\theta} * U, \pi_{\mathcal{D}^\theta} * V) \quad (4.4)$$

for any $U, V \in TM$, where $g_{\mathcal{D}}$ (resp. $g_{\mathcal{D}^\theta}$) denotes the Riemannian metric on $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$), $\pi_{\mathcal{D}}$ (resp. $\pi_{\mathcal{D}^\theta}$) is the projection operator of M to $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$) and f_1 is a certain positive differentiable function on $M_{\mathcal{D}^\theta}$. Now, we denote by $\widetilde{\nabla}$, ∇ , $\nabla^{\mathcal{D}}$ and $\nabla^{\mathcal{D}^\theta}$ the covariant differentiations with respect to $\widetilde{g}$, g , $g_{\mathcal{D}}$ and $g_{\mathcal{D}^\theta}$, respectively. Since we have from (1.6)

$$\begin{aligned} \nabla_X Y &= \nabla^{\mathcal{D}}_X Y - f_1^2 e^{f_1^2} (d_1 \log f_1)^* g_{\mathcal{D}}(X, Y), \\ \nabla_X Z &= \nabla_Z X = f_1^2 (Z \log f_1) X, \\ \nabla_Z W &= \nabla^{\mathcal{D}^\theta}_Z W \end{aligned} \quad (4.5)$$

for any $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$, where we put $(d_1 \log f_1)$ is the differential of $\log f_1$ with respect to $g_{\mathcal{D}^\theta}$.

Using Gauss formula and the above equation, we obtain

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla^{\mathcal{D}}_Y X - f_1^2 e^{f_1^2} (d_1 \log f_1)^* g_{\mathcal{D}}(X, Y) + \sigma(X, Y), \\ \widetilde{\nabla}_X Z &= \widetilde{\nabla}_Z X = f_1^2 (Z \log f_1) X + \sigma(X, Z), \\ \widetilde{\nabla}_Z W &= \nabla^{\mathcal{D}^\theta}_Z W + \sigma(Z, W) \end{aligned} \quad (4.6)$$

for any $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$.

Due to (4.6) between the Riemannian curvature tensors

- $R(U_1, U_2, U_3, U_4)$ with respect to g ,
- $R^{\mathcal{D}}(X_1, X_2, X_3, X_4)$ with respect to $g_{\mathcal{D}}$, and
- $R^{\mathcal{D}^\theta}(Z_1, Z_2, Z_3, Z_4)$ with respect to $g_{\mathcal{D}^\theta}$,

we know the following relations:

$$\begin{aligned}
R(X_1, X_2, X_3, X_4) &= e^{f_1^2} [R^{\mathcal{D}}(X_1, X_2, X_3, X_4) \\
&\quad - f_1^4 e^{f_1^2} \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2 \{g_{\mathcal{D}}(X_1, X_4)g_{\mathcal{D}}(X_2, X_3) \\
&\quad - g_{\mathcal{D}}(X_1, X_3)g_{\mathcal{D}}(X_2, X_4)\}], \\
R(X_1, Z_1, Z_2, X_2) &= -f_1^2 e^{f_1^2} \{(2 + f_1^2)(Z_1 \log f_1)(Z_2 \log f_1) \\
&\quad + \nabla^{\mathcal{D}^\theta}_{Z_1} \nabla^{\mathcal{D}^\theta}_{Z_2} \log f_1\} g_{\mathcal{D}}(X_1, X_2), \\
R(Z_1, Z_2, Z_3, Z_4) &= R^{\mathcal{D}^\theta}(Z_1, Z_2, Z_3, Z_4), \\
\text{Others} &= 0,
\end{aligned} \tag{4.7}$$

for any $X_1, X_2, X_3, X_4 \in \mathcal{D}$ and $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$.

By virtue of the above equation and the Gauss equation, we have the following

$$\begin{aligned}
\tilde{R}(X_1, X_2, X_3, X_4) &= e^{f_1^2} [R^{\mathcal{D}}(X_1, X_2, X_3, X_4) \\
&\quad - f_1^4 e^{f_1^2} \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2 \{g_{\mathcal{D}}(X_1, X_4)g_{\mathcal{D}}(X_2, X_3) \\
&\quad - g_{\mathcal{D}}(X_1, X_3)g_{\mathcal{D}}(X_2, X_4)\}] + \tilde{g}(\sigma(X_1, X_4), \sigma(X_2, X_3)) \\
&\quad - \tilde{g}(\sigma(X_1, X_3), \sigma(X_2, X_4)), \\
\tilde{R}(X_1, X_2, X_3, Z_1) &= \tilde{g}(\sigma(X_1, Z_1), \sigma(X_2, X_3)) - \\
&\quad - \tilde{g}(\sigma(X_1, X_3), \sigma(X_2, Z_1)), \\
\tilde{R}(X_1, X_2, Z_1, Z_2) &= \tilde{g}(\sigma(X_1, Z_2), \sigma(X_2, Z_1)) - \\
&\quad - \tilde{g}(\sigma(X_1, Z_1), \sigma(X_2, Z_2)), \\
\tilde{R}(X_1, Z_1, Z_2, Z_3) &= \tilde{g}(\sigma(X_1, Z_3), \sigma(Z_1, Z_2)) - \\
&\quad - \tilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, Z_3)), \\
\tilde{R}(X_1, Z_1, Z_2, X_2) &= -f_1^2 e^{f_1^2} \{(2 + f_1^2)(Z_1 \log f_1)(Z_2 \log f_1) \\
&\quad + \nabla^{\mathcal{D}^\theta}_{Z_1} \nabla^{\mathcal{D}^\theta}_{Z_2} \log f_1\} g_{\mathcal{D}}(X_1, X_2) \\
&\quad + \tilde{g}(\sigma(X_1, X_2), \sigma(Z_1, Z_2)) - \tilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, X_2)), \\
\tilde{R}(Z_1, Z_2, Z_3, Z_4) &= R^{\mathcal{D}^\theta}(Z_1, Z_2, Z_3, Z_4) + \tilde{g}(\sigma(Z_1, Z_4), \sigma(Z_2, Z_3)) \\
&\quad - \tilde{g}(\sigma(Z_1, Z_3), \sigma(Z_2, Z_4)),
\end{aligned} \tag{4.8}$$

for any $X_1, X_2, X_3, X_4 \in \mathcal{D}$ and $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$.

Next, we assume that our ambient manifold is an l.c.K.-space form. Then the curvature tensor $\tilde{R}$ satisfies (1.2). Using this, we can separate the curvature tensor $\tilde{R}$ as

$$\begin{aligned}
4\tilde{R}(X_1, X_2, X_3, X_4) &= c\{\tilde{g}(X_1, X_4)\tilde{g}(X_2, X_3) - \tilde{g}(X_1, X_3)\tilde{g}(X_2, X_4) + \\
&\quad + \tilde{g}(TX_1, X_4)\tilde{g}(TX_2, X_3) - \tilde{g}(TX_1, X_3)\tilde{g}(TX_2, X_4) - \\
&\quad - 2\tilde{g}(TX_1, X_2)\tilde{g}(TX_3, X_4)\} + \\
&\quad + 3\{P(X_1, X_4)\tilde{g}(X_2, X_3) - P(X_1, X_3)\tilde{g}(X_2, X_4) + \\
&\quad + P(X_2, X_3)\tilde{g}(X_1, X_4) - P(X_2, X_4)\tilde{g}(X_1, X_3)\} - \\
&\quad - \tilde{P}(X_1, X_4)\tilde{g}(TX_2, X_3) + \tilde{P}(X_1, X_3)\tilde{g}(TX_2, X_4) - \\
&\quad - \tilde{P}(X_2, X_3)\tilde{g}(TX_1, X_4) + \tilde{P}(X_2, X_4)\tilde{g}(TX_1, X_3) + \\
&\quad + 2\{\tilde{P}(X_1, X_2)\tilde{g}(TX_3, X_4) + \tilde{P}(X_3, X_4)\tilde{g}(TX_1, X_2)\}, \\
4\tilde{R}(X_1, X_2, X_3, Z_1) &= 3\{P(X_1, Z_1)\tilde{g}(X_2, X_3) - P(X_2, Z_1)\tilde{g}(X_1, X_3)\} \\
&\quad - \tilde{P}(X_1, Z_1)\tilde{g}(TX_2, X_3) + \tilde{P}(X_2, Z_1)\tilde{g}(TX_1, X_3) \\
&\quad + 2\tilde{P}(X_3, Z_1)\tilde{g}(TX_1, X_2), \\
2\tilde{R}(X_1, X_2, Z_1, Z_2) &= -c\tilde{g}(TX_1, X_2)\tilde{g}(TZ_1, Z_2) \\
&\quad + \tilde{P}(X_1, X_2)\tilde{g}(TZ_1, Z_2) + \tilde{P}(Z_1, Z_2)\tilde{g}(TX_1, X_2), \tag{4.9} \\
4\tilde{R}(X_1, Z_1, Z_2, X_2) &= c\{\tilde{g}(X_1, X_2)\tilde{g}(Z_1, Z_2) + \tilde{g}(TX_1, X_2)\tilde{g}(TZ_1, Z_2)\} \\
&\quad + 3\{P(X_1, X_2)\tilde{g}(Z_1, Z_2) + P(Z_1, Z_2)\tilde{g}(X_1, X_2)\} \\
&\quad - \tilde{P}(X_1, X_2)\tilde{g}(TZ_1, Z_2) - \tilde{P}(Z_1, Z_2)\tilde{g}(TX_1, X_2), \\
4\tilde{R}(X_1, Z_1, Z_2, Z_3) &= 3\{P(X_1, Z_3)\tilde{g}(Z_1, Z_2) - P(X_1, Z_2)\tilde{g}(Z_1, Z_3)\} \\
&\quad - \tilde{P}(X_1, Z_3)\tilde{g}(TZ_1, Z_2) + \tilde{P}(X_1, Z_2)\tilde{g}(TZ_1, Z_3) \\
&\quad + 2\tilde{P}(X_1, Z_1)\tilde{g}(TZ_2, Z_3), \\
4\tilde{R}(Z_1, Z_2, Z_3, Z_4) &= c\{\tilde{g}(Z_1, Z_4)\tilde{g}(Z_2, Z_3) - \tilde{g}(Z_1, Z_3)\tilde{g}(Z_2, Z_4) + \\
&\quad + \tilde{g}(TZ_1, Z_4)\tilde{g}(TZ_2, Z_3) - \tilde{g}(TZ_1, Z_3)\tilde{g}(TZ_2, Z_4) - \\
&\quad - 2\tilde{g}(TZ_1, Z_2)\tilde{g}(TZ_3, Z_4)\} + \\
&\quad + 3\{P(Z_1, Z_4)\tilde{g}(Z_2, Z_3) - P(Z_1, Z_3)\tilde{g}(Z_2, Z_4) - \\
&\quad + P(Z_2, Z_3)\tilde{g}(Z_1, Z_4)\} - \\
&\quad - \tilde{P}(Z_1, Z_4)\tilde{g}(TZ_2, Z_3) + \tilde{P}(Z_1, Z_3)\tilde{g}(TZ_2, Z_4) -
\end{aligned}$$

$$\begin{aligned}
& -\tilde{P}(Z_2, Z_3) \tilde{g}(TZ_1, Z_4) + \tilde{P}(Z_2, Z_4) \tilde{g}(TZ_1, Z_3) + \\
& + 2\{\tilde{P}(Z_1, Z_2) \tilde{g}(TZ_3, Z_4) + \tilde{P}(Z_3, Z_4) \tilde{g}(TZ_1, Z_2)\}
\end{aligned}$$

for any $X_1, X_2, X_3, X_4 \in \mathcal{D}$ and $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$.

Thus we have from (4.8) and (4.9) that

$$\begin{aligned}
& 4\{\tilde{g}(\sigma(X_1, X_4), \sigma(X_2, X_3)) - \tilde{g}(\sigma(X_1, X_3), \sigma(X_2, X_4))\} = \\
& - 4e^{f_1^2} R^{\mathcal{D}}(X_1, X_2, X_3, X_4) \\
& + \{c + 4f_1^2 \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2\} \{\tilde{g}(X_1, X_4) \tilde{g}(X_2, X_3) \\
& - \tilde{g}(X_1, X_3) \tilde{g}(X_2, X_4)\} + c\{\tilde{g}(TX_1, X_4) \tilde{g}(TX_2, X_3) \\
& - \tilde{g}(TX_1, X_3) \tilde{g}(TX_2, X_4) - 2\tilde{g}(TX_1, X_2) \tilde{g}(TX_3, X_4)\} \\
& + 3\{P(X_1, X_4) \tilde{g}(X_2, X_3) - P(X_1, X_3) \tilde{g}(X_2, X_4) \\
& + P(X_2, X_3) \tilde{g}(X_1, X_4) - P(X_2, X_4) \tilde{g}(X_1, X_3)\} \\
& - \tilde{P}(X_1, X_4) \tilde{g}(TX_2, X_3) + \tilde{P}(X_1, X_3) \tilde{g}(TX_2, X_4) \\
& - \tilde{P}(X_2, X_3) \tilde{g}(TX_1, X_4) + \tilde{P}(X_2, X_4) \tilde{g}(TX_1, X_3) \\
& + 2\{\tilde{P}(X_1, X_2) \tilde{g}(TX_3, X_4) + \tilde{P}(X_3, X_4) \tilde{g}(TX_1, X_2)\}, \\
& 4\{\tilde{g}(\sigma(X_1, Z_1), \sigma(X_2, X_3)) - \tilde{g}(\sigma(X_1, X_3), \sigma(X_2, Z_1))\} = \\
& 3\{P(X_1, Z_1) \tilde{g}(X_2, X_3) - P(X_2, Z_1) \tilde{g}(X_1, X_3)\} \tag{4.10} \\
& - \tilde{P}(X_1, Z_1) \tilde{g}(TX_2, X_3) + \tilde{P}(X_2, Z_1) \tilde{g}(TX_1, X_3) \\
& + 2\tilde{P}(X_3, Z_1) \tilde{g}(TX_1, X_2), \\
& 2\{\tilde{g}(\sigma(X_1, Z_2), \sigma(X_2, Z_1)) - \tilde{g}(\sigma(X_1, Z_1), \sigma(X_2, Z_2))\} = \\
& - c\tilde{g}(TX_1, X_2) \tilde{g}(TZ_1, Z_2) + \tilde{P}(X_1, X_2) \tilde{g}(TZ_1, Z_2) \\
& + \tilde{P}(Z_1, Z_2) \tilde{g}(TX_1, X_2), \\
& 4\{\tilde{g}(\sigma(X_1, X_2), \sigma(Z_1, Z_2)) - \tilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, X_2))\} = \\
& 4f_1^2 e^{f_1^2} \{(2 + f_1^2)(Z_1 \log f_1)(Z_2 \log f_1) \\
& + \nabla^{\mathcal{D}^\theta}{}_{Z_1} \nabla^{\mathcal{D}^\theta}{}_{Z_2} \log f_1\} g_{\mathcal{D}}(X_1, X_2) \\
& + c\{\tilde{g}(X_1, X_2) \tilde{g}(Z_1, Z_2) + \tilde{g}(TX_1, X_2) \tilde{g}(TZ_1, Z_2)\} \\
& + 3\{P(X_1, X_2) \tilde{g}(Z_1, Z_2) + P(Z_1, Z_2) \tilde{g}(X_1, X_2)\} \\
& - \tilde{P}(X_1, X_2) \tilde{g}(TZ_1, Z_2) - \tilde{P}(Z_1, Z_2) \tilde{g}(TX_1, X_2), \\
& 4\{\tilde{g}(\sigma(X_1, Z_3), \sigma(Z_1, Z_2)) - \tilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, Z_3))\} = \\
& 3\{P(X_1, Z_3) \tilde{g}(Z_1, Z_2) - P(X_1, Z_2) \tilde{g}(Z_1, Z_3)\}
\end{aligned}$$

$$\begin{aligned}
& -\tilde{P}(X_1, Z_3)\tilde{g}(TZ_1, Z_2) + \tilde{P}(X_1, Z_2)\tilde{g}(TZ_1, Z_3) \\
& + 2\tilde{P}(X_1, Z_1)\tilde{g}(TZ_2, Z_3), \\
& 4\{\tilde{g}(\sigma(Z_1, Z_4), \sigma(Z_2, Z_3)) - \tilde{g}(\sigma(Z_1, Z_3), \sigma(Z_2, Z_4))\} = \\
& -4R^{\mathcal{D}^\theta}(Z_1, Z_2, Z_3, Z_4) \\
& + c\{\tilde{g}(Z_1, Z_4)\tilde{g}(Z_2, Z_3) - \tilde{g}(Z_1, Z_3)\tilde{g}(Z_2, Z_4) \\
& + \tilde{g}(TZ_1, Z_4)\tilde{g}(TZ_2, Z_3) - \tilde{g}(TZ_1, Z_3)\tilde{g}(TZ_2, Z_4) \\
& - 2\tilde{g}(TZ_1, Z_2)\tilde{g}(TZ_3, Z_4)\} + 3\{P(Z_1, Z_4)\tilde{g}(Z_2, Z_3) \\
& - P(Z_1, Z_3)\tilde{g}(Z_2, Z_4) + P(Z_2, Z_3)\tilde{g}(Z_1, Z_4) \\
& P(Z_2, Z_4)\tilde{g}(Z_1, Z_3) - \tilde{P}(Z_1, Z_4)\tilde{g}(TZ_2, Z_3) \\
& + \tilde{P}(Z_1, Z_3)\tilde{g}(TZ_2, Z_4) - \tilde{P}(Z_2, Z_3)\tilde{g}(TZ_1, Z_4) \\
& + \tilde{P}(Z_2, Z_4)\tilde{g}(TZ_1, Z_3) + 2\{\tilde{P}(Z_1, Z_2)\tilde{g}(TZ_3, Z_4) \\
& + \tilde{P}(Z_3, Z_4)\tilde{g}(TZ_1, Z_2)\},
\end{aligned}$$

for any $X_1, X_2, X_3, X_4 \in \mathcal{D}$ and $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$. Let

$$\begin{aligned}
& e_1, \dots, e_p, e_1^*, \dots, e_p^*, \\
& e_{2p+1}, \dots, e_{2p+q}, e_{2p+1}^*, \dots, e_{2p+q}^*, \\
& e_{n+q+1}, \dots, e_{n+q+s}, e_{n+q+1}^*, \dots, e_{n+q+s}^*
\end{aligned}$$

be a generalized adapted local frame of $\widetilde{M}$, [4].

Using this frame, the Gauss equation (4.10) is written as

$$\begin{aligned}
& 4\{\tilde{g}(\sigma_{kh}, \sigma_{ji}) - \tilde{g}(\sigma_{ki}, \sigma_{jh})\} = \\
& -4ef_1^2 R^{\mathcal{D}}_{kjih} + f_1^2 \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2 (\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh}) \\
& - 3(P_{kh}\delta_{ji} - P_{ki}\delta_{jh} + P_{ji}\delta_{kh} - P_{jh}\delta_{ki}) \\
& + P(Je_k, e_h)\tilde{g}(Je_j, e_i) - P(Je_k, e_i)\tilde{g}(Je_j, e_h) \\
& + P(Je_j, e_i)\tilde{g}(Je_k, e_h) - P(Je_j, e_h)\tilde{g}(Je_k, e_i) \\
& - 2\{P(Je_k, e_j)\tilde{g}(Je_i, e_h) + P(Je_i, e_h)\tilde{g}(Je_k, e_j)\}, \\
& 4\{\tilde{g}(\sigma_{ja}, \sigma_{ih}) - \tilde{g}(\sigma_{jh}, \sigma_{ia})\} = 3(P_{ja}\delta_{ih} - P_{ia}\delta_{jh}) \\
& - P(Je_j, e_a)\tilde{g}(Je_i, e_h) + P(Je_j, e_a)\tilde{g}(Je_j, e_h) \\
& + 2P(Je_h, e_a)\tilde{g}(Je_j, e_i), \\
& 2\{\tilde{g}(\sigma_{ia}, \sigma_{hb}) - \tilde{g}(\sigma_{ib}, \sigma_{ha})\} = -c\tilde{g}(Je_i, e_h)\tilde{g}(Te_b, e_a) \\
& + P(Je_b, e_a)\tilde{g}(Je_i, e_h) + P(Je_i, e_h)\tilde{g}(Te_b, e_a),
\end{aligned}$$

$$\begin{aligned}
4\{\tilde{g}(\sigma_{ih}, \sigma_{ba}) - \tilde{g}(\sigma_{ia}, \sigma_{bh})\} &= 4f_1^2\{(2 + f_1^2)(e_b \log f_1)(e_a \log f_1) \\
&\quad + \nabla^{\mathcal{D}^\theta}_{e_b} \nabla^{\mathcal{D}^\theta}_{e_a} \log f_1\} \delta_{ih} + c\{\delta_{ih} \delta_{ba} + \tilde{g}(Je_i, e_h) \tilde{g}(Te_b, e_a)\} \\
3(P_{ih} \delta_{ba} + P_{ba} \delta_{ih}) - P(Je_i, e_h) \tilde{g}(Te_b, e_a) \\
&\quad - P(Je_b, e_a) \tilde{g}(Je_i, e_h), \tag{4.11}
\end{aligned}$$

$$\begin{aligned}
4\{\tilde{g}(\sigma_{ha}, \sigma_{cb}) - \tilde{g}(\sigma_{hb}, \sigma_{ca})\} &= 3(P_{ha} \delta_{cb} - P_{hb} \delta_{ca}) \\
&\quad - P(Je_h, e_a) \tilde{g}(Te_c, e_b) + P(Je_h, e_b) \tilde{g}(Te_c, e_a) \\
&\quad + 2P(Je_h, e_c) \tilde{g}(Te_b, e_a),
\end{aligned}$$

$$\begin{aligned}
4\{\tilde{g}(\sigma_{da}, \sigma_{cb}) - \tilde{g}(\sigma_{db}, \sigma_{ch})\} &= -4R^{\mathcal{D}^\theta}_{dcba} + c\{\delta_{da} \delta_{cb} - \delta_{db} \delta_{ca} \\
&\quad + \tilde{g}(Te_d, e_a) \tilde{g}(Te_c, e_b) - \tilde{g}(Te_d, e_b) \tilde{g}(Te_c, e_a) \\
&\quad - 2\tilde{g}(Te_d, e_c) \tilde{g}(Te_b, e_a)\} \\
&\quad + 3(P_{da} \delta_{cb} - P_{db} \delta_{ca} + P_{cb} \delta_{da} - P_{ca} \delta_{db}) \\
&\quad - P(Je_d, e_a) \tilde{g}(Te_c, e_b) + P(Je_d, e_b) \tilde{g}(Te_c, e_a) \\
&\quad - P(Je_c, e_b) \tilde{g}(Te_d, e_a) + P(Je_c, e_a) \tilde{g}(Te_d, e_b) \\
&\quad + 2\{P(Je_d, e_c) \tilde{g}(Te_b, e_a) + P(Je_b, e_a) \tilde{g}(Te_d, e_c)\},
\end{aligned}$$

for any

$$k, j, i, h \in \{1, 2, \dots, 2p\}$$

and

$$d, c, b, a \in \{2p+1, 2p+2, \dots, 2p+q\},$$

where we put $\sigma(e_\mu, e_\lambda) = \sigma_{\mu\lambda}$, etc.

The mean curvature vector H and the mean curvature $\|H\|$ are respectively given by

$$H = \frac{1}{n} \sum_{\mu=1}^n \sigma_{\mu\mu}, \quad \|H\|^2 = \frac{1}{n^2} \sum_{\mu, \lambda=1}^n \tilde{g}(\sigma_{\mu\mu}, \sigma_{\lambda\lambda}) \tag{4.12}$$

and the length $\|\sigma\|$ of the second fundamental form σ is given by

$$\|\sigma\|^2 = \sum_{\mu, \lambda=1}^n \tilde{g}(\sigma_{\mu\lambda}, \sigma_{\mu\lambda}) = \sum_{\mu, \lambda=1}^n \sum_{r=n+1}^m \{\tilde{g}(\sigma_{\mu\lambda}, e_r)\}^2 \tag{4.13}$$

for any local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ of $T\widetilde{M}$.

By virtue of the Gauss equations, we have

$$\sum_{\mu, \lambda=1}^n (R_{\mu\lambda\mu\lambda} - \widetilde{R}_{\mu\lambda\mu\lambda}) = \|\sigma\|^2 - n\|H\|^2. \tag{4.14}$$

On the other hand, we have from (4.10) that

$$\begin{aligned}
4 \sum_{\mu, \lambda=1}^n \tilde{R}_{\mu\lambda\mu\lambda} &= -(n^2 + 2n - 3q)c - 6(n-2) \sum_{\mu=1}^n P_{\mu\mu} \\
&\quad - 6 \sum_{b=2p+1}^{2p+q} P_{bb} - 3c \sum_{b,a=2p+1}^{2p+q} T_{ba}T_{ba} + 6 \sum_{b,a=2p+1}^{2p+q} P(Je_b, e_a)T_{ba},
\end{aligned} \tag{4.15}$$

where $T_{ba} = \tilde{g}(T_b^c e_c, e_a)$ for any $c, b, a \in \{2p+1, 2p+2, \dots, 2p+q = n\}$. We know T_{ba} is skew-symmetric.

Moreover, we have from (1.9) and (4.7) that

$$\begin{aligned}
4 \sum_{\mu, \lambda=1}^n R_{\mu\lambda\mu\lambda} &= -(e^{f_1^2} \tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + 8pf_1^2 \{(2p-1)f_1^2 \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2 + \\
&\quad + 2(2 + f_1^2) \sum_{a=2p+1}^{2p+q} (e_a \log f_1)^2 + 2 \sum_{a=2p+1}^{2p+q} \nabla^{\mathcal{D}^\theta} e_a \nabla^{\mathcal{D}^\theta} e_a \log f_1\},
\end{aligned} \tag{4.16}$$

where $\tau^{\mathcal{D}}$ (resp. $\tau^{\mathcal{D}^\theta}$) denotes the scalar curvature with respect to $g_{\mathcal{D}}$ (resp. $g_{\mathcal{D}^\theta}$).

Substituting (4.15) and (4.16) into (4.14), we obtain

$$\begin{aligned}
4\|\sigma\|^2 &= 4n\|H\|^2 + 8pf\{(2p-1)f_1^2 \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2 + \\
&\quad + 2(2 + f_1^2) \sum_{a=2p+1}^{2p+q} (e_a \log f_1)^2\} + (n^2 + 2n - 3q)c + \\
&\quad + 3c \sum_{b,a=2p+1}^{2p+q} (T_{ba})^2 - 4(e^{f_1^2} \tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\
&\quad + 16pf_1^2 \sum_{b,a=2p+1}^{2p+q} \nabla_{e_a}^{\mathcal{D}^\theta} \nabla_{e_a}^{\mathcal{D}^\theta} \log f_1 + 6(n-2) \sum_{\mu=1}^n P_{\mu\mu} + \\
&\quad + 6 \sum_{a=2p+1}^{2p+q} P_{aa} - 6 \sum_{b,a=2p+1}^{2p+q} P(Je_b, e_a)T_{ba}.
\end{aligned} \tag{4.17}$$

Thus we have

Theorem 4.1. *In a first type warped product semi-slant submanifold in an l.c.K.-space form, the mean curvature satisfies the inequality*

$$\begin{aligned}
& 4n\|H\|^2 + 8pf_1^2\{(2p-1)f_1^2\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2 \\
& + 2(2+f_1^2)\sum_{a=2p+1}^{2p+q}(e_a\log f_1)^2\} + (n^2+2n-3q)c \\
& + 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f_1^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) \\
& + 16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla_{e_a}^{\mathcal{D}^\theta}\nabla_{e_a}^{\mathcal{D}^\theta}\log f_1 + 6(n-2)\sum_{\mu=1}^n P_{\mu\mu} \\
& + 6\sum_{a=2p+1}^{2p+q} P_{aa} - 6\sum_{b,a=2p+1}^{2p+q} P(Je_b, e_a)T_{ba} \geq 0.
\end{aligned} \tag{4.18}$$

Corollary 4.2. *Under the same condition with Theorem 4.1, the equality case of (4.18) is that the submanifold is locally totally geodesic and the warping function f_1 satisfies*

$$\begin{aligned}
& 8pf_1^2\{(2p-1)f_1^2\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2 + \\
& + 2(2+f_1^2)\sum_{a=2p+1}^{2p+q}(e_a\log f_1)^2\} + (n^2+2n-3q)c + \\
& + 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f_1^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\
& + 16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla_{e_a}^{\mathcal{D}^\theta}\nabla_{e_a}^{\mathcal{D}^\theta}\log f_1 + 6(n-2)\sum_{\mu=1}^n P_{\mu\mu} + \\
& + 6\sum_{a=2p+1}^{2p+q} P_{aa} - 6\sum_{b,a=2p+1}^{2p+q} P(Je_b, e_a)T_{ba} = 0.
\end{aligned}$$

and

$$\begin{aligned}
& (n^2+2n-3q)c + 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f_1^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\
& 16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla_{e_a}^{\mathcal{D}^\theta}\nabla_{e_a}^{\mathcal{D}^\theta}\log f_1 + 6(n-2)\sum_{\mu=1}^n P_{\mu\mu} +
\end{aligned}$$

$$+ 6 \sum_{a=2p+1}^{2p+q} P_{aa} - 6 \sum_{b,a=2p+1}^{2p+q} P(Je_b, e_a)T_{ba} \leq 0.$$

5. ACKNOWLEDGEMENTS

The author is thankful to the referee for his/her valuable comments which have improved the manuscript.

REREFENCES

- [1] Aurel Bejancu. CR submanifolds of a Kaehler manifold. I. *Proc. Amer. Math. Soc.*, 69(1):135–142, 1978.
- [2] Aurel Bejancu. CR submanifolds of a Kaehler manifold. II. *Trans. Amer. Math. Soc.*, 250:333–345, 1979.
- [3] Aurel Bejancu. *Geometry of CR-submanifolds*, volume 23 of *Mathematics and its Applications (East European Series)*. D. Reidel Publishing Co., Dordrecht, 1986.
- [4] Vittoria Bonanzinga, Koji Matsumoto. Semi-slant submanifolds in locally conformal Kaehler manifolds. *to appear*.
- [5] Vittoria Bonanzinga, Koji Matsumoto. Warped product CR-submanifolds in locally conformal Kaehler manifolds. *Period. Math. Hungar.*, 48(1-2):207–221, 2004.
- [6] Bang-Yen Chen. *Geometry of submanifolds*. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 22.
- [7] Bang-Yen Chen. CR-submanifolds of a Kaehler manifold. I. *J. Differential Geom.*, 16(2):305–322, 1981.
- [8] Bang-Yen Chen. CR-submanifolds of a Kaehler manifold. II. *J. Differential Geom.*, 16(3):493–509 (1982), 1981.
- [9] Bang-Yen Chen. *Geometry of slant submanifolds*. Katholieke Universiteit Leuven, Louvain, 1990.
- [10] Toyoko Kashiwada. Some properties of locally conformal Kähler manifolds. *Hokkaido Math. J.*, 8(2):191–198, 1979.
- [11] Viqar Azam Khan, Meraj Ali Khan. Semi-slant submanifolds of a nearly Kaehler manifold. *Turkish J. Math.*, 31(4):341–353, 2007.
- [12] Barrett O’Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. With applications to relativity.
- [13] Neculai Papaghiuc. Semi-slant submanifolds of a Kaehlerian manifold. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat.*, 40(1):55–61, 1994.
- [14] Izu Vaisman. On locally conformal almost Kähler manifolds. *Israel Journal of Mathematics*, 24(3-4):338–351, Dec 1976.

Received: July 3, 2017, accepted: September 2, 2017.

Koji Matsumoto

2-3-65 NISHI-ODORI, YONEZAWA, YAMAGATA, 992-0059, JAPAN

Email: tokiko_matsumoto@yahoo.com