Some remarks concerning strongly separately continuous functions on spaces $\ell_p$ with $p \in [1, +\infty]$

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Abstract. We give a sufficient condition on strongly separately continuous function $f$ to be continuous on space $\ell_p$ for $p \in [1, +\infty]$. We prove the existence of an ssc function $f : \ell_\infty \to \mathbb{R}$ which is not Baire measurable. We show that any open set in $\ell_p$ is the set of discontinuities of a strongly separately continuous real-valued function for $p \in [1, +\infty]$.

1. Introduction

The notion of real-valued strongly separately continuous (ssc) function defined on $\mathbb{R}^n$ was introduced and studied by Dzagnidze in his paper [1]. Later, the authors extended in [7] the notion of the strong separate continuity to functions defined on the Hilbert space $\ell_2$ equipped with the norm topology and proved, in particular, that there exists a real-valued ssc function on $\ell_2$ which is everywhere discontinuous. Visnyai [8] constructed an ssc function $f : \ell_2 \to \mathbb{R}$ which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, he gave a sufficient condition for a strongly separately continuous function to be continuous on $\ell_2$.

In [3] Karlova extended the concept of an ssc function on any $S$-open subset of a product of topological spaces and investigated ssc functions with open set of discontinuities defined on a special subsets of a product of a sequence of normed spaces. Karlova and Mykhaylyuk obtained a characterization of the set of all points of discontinuity of strongly separately continuous functions defined on subspaces of products of finite-dimensional normed spaces [4].

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Karlova and Visnyai proved in [5] that any open set in $\ell_p$ is the set of discontinuities of a strongly separately continuous real-valued function for $p \in [1, +\infty)$ (see [5, Theorem 4.1]). Unfortunately, the proof of this result contains a gap, which we remove in Theorem 5.3 of this paper.

The Baire classification of ssc functions was investigated in [3] and [5]. It was proved that for every $2 \leq \alpha < \omega_1$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ which belongs the $\alpha$‘th Baire class and does not belong to the $\beta$‘th Baire class on $\ell_p$ for $\beta < \alpha$, $p \in [1, +\infty)$.

In this paper we continue to study ssc functions defined on spaces $\ell_p$ with $p \in [1, +\infty]$. In Section 3 we give a sufficient condition on ssc function $f$ defined on $\ell_p$ to be continuous. Further, we prove in Section 4 that there exists an ssc function $f : \ell_\infty \to \mathbb{R}$ which is not Baire measurable. Section 5 contains a result on a construction of ssc functions with open set of discontinuities.

2. Definitions and notations

We denote by $\ell_p$, $p \in [1, +\infty)$, the normed space consisting of all sequences $x = (x_k)_{k=1}^\infty$ of reals such that $\sum_{k=1}^\infty |x_k|^p < +\infty$ endowed with the standard norm $\|x\|_p$ defined by the rule

$$\|x\|_p = \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p}$$

for all $x = (x_k)_{k=1}^\infty \in \ell_p$.

Let $\ell_\infty$ be the space of all bounded sequences of reals with the norm

$$\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$$

for all $x = (x_k)_{k=1}^\infty \in \ell_\infty$.

If $p \in [1, +\infty]$, $x^0 \in \ell_p$ and $\delta > 0$, then we write

$$B_p(x^0, \delta) = \{x \in \ell_p : \|x - x^0\|_p < \delta\}.$$

**Definition 2.1.** Let $p \in [1, +\infty]$, $x^0 = (x^0_k)_{k=1}^\infty \in \ell_p$ and $(Y, | \cdot - \cdot |)$ be a metric space. A function $f : \ell_p \to Y$ is said to be

- **separately continuous at a point** $x^0$ with respect to the $k$-th variable if the function $\varphi_k : \mathbb{R} \to Y$, $\varphi_k(t) = f(x^0_1, \ldots, x^0_{k-1}, t, x^0_{k+1}, \ldots)$ for all $t \in \mathbb{R}$, is continuous at $x^0_k$.

- **strongly separately continuous at a point** $x^0$ with respect to the $k$-th variable if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x = (x_k)_{k=1}^\infty \in B_p(x^0, \delta)$$
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\[ |f(x_1, \ldots, x_k, \ldots) - f(x_1, \ldots, x_{k-1}, x_k^0, x_{k+1}, \ldots)| < \varepsilon. \quad (2.1) \]

If \( f \) is strongly separately continuous at \( x^0 \) with respect to each variable, then \( f \) is said to be \emph{strongly separately continuous at} \( x^0 \). Moreover, \( f \) is \emph{(strongly) separately continuous on} \( \ell_p \) if it is (strongly) separately continuous at each point of \( \ell_p \).

It is easy to see that

continuity \(\Rightarrow\) strong separate continuity \(\Rightarrow\) separate continuity.

None of the converse implications is true as the following examples show.

**Example 2.2.** Let

\[ f(x_1, x_2, \ldots) = \begin{cases} \frac{x_1 \cdot x_2}{x_1^2 + x_2^2}, & x_1^2 + x_2^2 \neq 0, \\ 0, & \text{otherwise}. \end{cases} \]

The function \( f: \ell_p \to \mathbb{R} \) is separately continuous on \( \ell_p \) for every \( p \in [1, +\infty] \), but is not strongly separately continuous at \((0, 0, \ldots)\) for any \( p \in [1, +\infty] \) (see remarks after Theorem 3.1).

**Example 2.3.** Let \( A = \{ x = (x_k)_{k=1}^{\infty} \in \ell_p : |\{k : x_k \in \mathbb{Q}\}| < \aleph_0 \} \). We put

\[ f(x_1, x_2, \ldots) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise}. \end{cases} \]

The function \( f: \ell_p \to \mathbb{R} \) is strongly separately continuous on \( \ell_p \), but is everywhere discontinuous for every \( p \in [1, +\infty] \).

**Proof.** Fix \( p \in [1, +\infty] \). It is easy to see that both \( A \) and \( \ell_p \setminus A \) are everywhere dense in \( \ell_p \). This imply that \( f \) is everywhere discontinuous on \( \ell_p \). Moreover, if \( x \) and \( y \) differs in at most one coordinate, then \( x \in A \) if and only if \( y \in A \). Therefore, \( |f(x) - f(y)| = 0 \) and (2.1) holds.

\[ \square \]

### 3. Continuity of ssc functions

We will prove in this section the sufficient condition on strongly separately continuous functions to be continuous on spaces \( \ell_p \).

For \( p \in [1, +\infty] \), \( x, y \in \ell_p \) and \( n \in \mathbb{N} \) we put

\[ \gamma_p^n(x, y) = \begin{cases} \sum_{k>n} |x_k - y_k|^p, & p < +\infty, \\ \sup_{k>n} |x_k - y_k|, & p = +\infty. \end{cases} \]
Theorem 3.1. Let $p \in [1, +\infty]$, $x^0 = (x^0_k)_{k=1}^{\infty} \in \ell_p$, $(Y, |\cdot|)$ be a metric space and $f: \ell_p \to Y$ be a strongly separately continuous function at $x^0$. If for every $\varepsilon > 0$ there exist $\delta > 0$ and $K \in \mathbb{N}$ such that
\[ \gamma^K_p(x^0, y) < \delta \Rightarrow |f(y_1, \ldots) - f(y_1, \ldots, y_K, x^0_{K+1}, \ldots)| < \varepsilon \] (3.1)
for all $y = (y_k)_{k=1}^{\infty} \in \ell_p$, then $f$ is continuous at $x^0$.

Proof. Fix $\varepsilon > 0$. According to the assumption there exists $\delta_0 > 0$ and $K \in \mathbb{N}$ such that the inequality
\[ \gamma^K_p(x^0, y) < \delta_0 \]
implies the inequality
\[ |f(y_1, y_2, \ldots) - f(y_1, \ldots, y_K, x^0_{K+1}, x^0_{K+2}, \ldots)| < \frac{\varepsilon}{2} \]
for all $y \in \ell_p$. Since $f$ is strongly separately continuous at the point $x^0$, for every $k \in \{1, 2, \ldots, K\}$ there exists $\delta_k > 0$ such that
\[ |f(x_1, \ldots, x_k, \ldots) - f(x_1, \ldots, x_{k-1}, x^0_k, x^0_{k+1}, \ldots)| < \frac{\varepsilon}{2K} \]
for all $x \in B_p(x^0, \delta_k)$. We put
\[ \delta = \begin{cases} \min \{ \sqrt[p]{\delta_0}, \delta_1, \ldots, \delta_K \} , & p < \infty \\ \min \{ \delta_0, \delta_1, \ldots, \delta_K \} , & p = \infty. \end{cases} \]
Let us take $x = (x_k)_{k=1}^{\infty} \in B_p(x^0, \delta)$ and observe that
\[ (x^0_1, \ldots, x^0_k, x^0_{k+1}, \ldots) \in B_p(x^0, \delta) \]
for every $k \in \{1, \ldots, K\}$. It follows that
\[ |f(x_1, x_2, \ldots) - f(x^0_1, x^0_2, \ldots)| \leq |f(x_1, x_2, \ldots) - f(x^0_1, x^0_2, \ldots)| + \\
+ |f(x^0_1, x_2, x_3, \ldots) - f(x^0_1, x^0_2, x^0_3, \ldots)| + \cdots + \\
+ |f(x^0_1, \ldots, x^0_{K-1}, x_K, x^0_{K+1}, \ldots) - f(x^0_1, \ldots, x^0_{K-1}, x^0_K, x^0_{K+1}, \ldots)| + \\
+ |f(x^0_1, \ldots, x^0_{K-1}, x^0_K, x^0_{K+1}, \ldots) - f(x^0_1, \ldots, x^0_K, x^0_{K+1}, x^0_{K+2}, \ldots)| < \\
< K \cdot \frac{\varepsilon}{2K} + \frac{\varepsilon}{2} = \varepsilon. \]

Hence, $f$ is continuous at $x^0$. \(\square\)

Now we are ready to show that the function $f$ from Example 2.2 is not strongly separately continuous at $x^0 = (0, 0, \ldots)$. Assume the contrary and observe that for $K = 2$ we have $|f(y_1, y_2, \ldots) - f(y_1, y_2, 0, \ldots)| = 0$ for all $y \in \ell_p$. It follows that condition (3.1) holds for any $\varepsilon > 0$ and for any $\delta > 0$. Therefore, $f$ has to be continuous at $x^0$ by Theorem 3.1, a contradiction.
As a straightforward corollary from Theorem 3.1 we obtain the next result.

**Theorem 3.2.** Let $p \in [1, +\infty]$, $(Y, |\cdot|)$ be a metric space and $f : \ell_p \to Y$ be a strongly separately continuous function. If

$$
\forall x \in \ell_p \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \exists K \in \mathbb{N}
$$

$$
|f(y_1, y_2, \ldots) - f(y_1, \ldots, y_K, x_{K+1}, x_{K+2}, \ldots)| < \varepsilon
$$

for all $y \in \ell_p$ with $\gamma^K_p(x, y) < \delta$, then $f$ is continuous on $\ell_p$.

### 4. Baire classification of ssc-functions

Let us recall the definition of Baire classes of functions. We denote the collection of all continuous maps $f : X \to Y$ between topological spaces $X$ and $Y$ by $B_0(X, Y)$. Assume that the classes $B_\xi(X, Y)$ are already defined for all $0 \leq \xi < \alpha$, where $\alpha < \omega_1$. Then $f : X \to Y$ is said to be of the $\alpha$-th Baire class, $f \in B_\alpha(X, Y)$, if $f$ is a pointwise limit of a sequence of maps $f_n \in B_\xi_n(X, Y)$, where $\xi_n < \alpha$.

Let $0 \leq \alpha < \omega_1$, $X$ be a metrizable space, $Y$ is a topological space and let $Z$ be a locally convex space. According to Rudin’s result [6] each map $f : X \times Y \to Z$, which is continuous with respect to the first variable and is of the $\alpha$-th Baire class with respect to the second one, belongs to the $(\alpha + 1)$-th Baire class on $X \times Y$. It is easy to prove the corollary of the Rudin Theorem (see [3, Proposition 3.1]): if $n \in \mathbb{N}$, $X_1, \ldots, X_n$ are metrizable spaces and $Z$ is a locally convex space, then every separately continuous map $f : \prod_{i=1}^n X_i \to Z$ belongs to the $(n - 1)$-th Baire class.

On the other hand, it was proved in [3, Corollary 2.8] that any strongly separately continuous map $f : \prod_{i=1}^n X_i \to Z$ is continuous. Therefore, it is interesting to study Baire classification of ssc functions defined on subsets of products of infinitely many factors, in particular, on spaces $\ell_p$.

**Definition 4.1.** A subset $A \subseteq X$ of a Cartesian product $X = \prod_{k=1}^\infty X_k$ of sets $X_1, X_2, \ldots$ is called $S$-open [3], if

$$
\{x = (x_k)_{k=1}^\infty \in X : |\{k : x_k \neq a_k\}| \leq 1\} \subseteq A
$$

for all $a = (a_k)_{k=1}^\infty \in A$.

Notice that any space $\ell_p$ as a subset of the set $\mathbb{R}^\omega$ of all sequences is an example of $S$-open set.

**Proposition 4.2.** For every $p \in [1, +\infty]$ there exists an $S$-open set $A \subseteq \ell_p$ which is not Borel measurable.
**Proof.** Firstly, we consider the case $p < +\infty$. Define a relation $\sim$ on $\ell_p$ in the following way:

$$x \sim y \iff \text{the set } \{k \in \mathbb{N} : x_k \neq y_k\} \text{ is finite}$$

for all $x = (x_k)_{k=1}^{\infty}, y = (y_k)_{k=1}^{\infty} \in \ell_p$. Clearly, $\sim$ defines the equivalence relation on $\ell_p$. Consider a partition $(\sigma_i : i \in I)$ of $\ell_p$ on the equivalence classes $\sigma_i$.

It is not hard to verify that $|I| = c$. Then there are $2^c$ many sets of the form $\bigcup_{i \in J} \sigma_i$, where $J \subseteq I$.

On the other hand, since $\ell_p$ is separable, it is a second countable space. Hence, the cardinality of the collection of all open subsets of $\ell_p$ is $c$. Therefore, the cardinality of the collection of all Borel measurable sets in $\ell_p$ is also equal to $c$. Consequently, there exists a set $J \subseteq I$ such that the union

$$A = \bigcup_{i \in J} \sigma_i$$

is not Borel measurable.

Let $a = (a_k)_{k=1}^{\infty} \in A$ and $x = (x_k)_{k=1}^{\infty} \in \ell_p$ be a sequence which differs from $a$ in at most one coordinate. Since $a \in \sigma_i$ for some $i \in J$, there exists a point $y = (y_k)_{k=1}^{\infty} \in \ell_p$ such that $\sigma_i = [y]$ and $|\{k \in \mathbb{N} : a_k \neq y_k\}| < \aleph_0$. Clearly, $|\{k \in \mathbb{N} : x_k \neq y_k\}| < \aleph_0$. Therefore, $x \in \sigma_i \subseteq A$. Hence, the set $A$ is $S$-open.

Now let $p = +\infty$. For every $r \in \mathbb{R}$ we write

$$B_r = \{x \in \ell_1 : \|x\|_1 \leq r\}$$

and show that $B_r$ is closed in $\ell_\infty$. Suppose that $\|x\|_1 = \sum_{k=1}^{\infty} |x_k| > r$. There exists a number $m \in \mathbb{N}$ such that

$$\sum_{k=1}^{m} |x_k| > r.$$

Since the map $s : \mathbb{R}^m \to \mathbb{R}, s(y_1, \ldots, y_m) = \sum_{k=1}^{m} |y_k|$ is continuous at $(x_1, \ldots, x_m)$, there exists $\delta > 0$ such that

$$|x_k - y_k| < \delta \text{ for every } k \in \{1, \ldots, m\} \implies \sum_{k=1}^{m} |y_k| > r.$$

Then

$$B_\infty(x, \delta) \subseteq \ell_\infty \setminus B_r.$$

Therefore, $\ell_\infty \setminus B_r$ is open in $\ell_\infty$ and hence $B_r$ is closed.

Now let $G$ be an open subset of $\ell_1$. Then there exists a sequence $(\delta_n)_{n=1}^{\infty}$ of reals and $(x_n)_{n=1}^{\infty}$ of points from $\ell_1$ such that $G = \bigcup_{n=1}^{\infty} B_1(x_n, \delta_n)$. It follows that $G$ is an $F_\sigma$-subset of $\ell_\infty$. Consequently, every Borel measurable subset of $\ell_1$ is Borel measurable in $\ell_\infty$. 
Conversely, if \( U = B_\infty(0,1) \cap \ell_1 = \{x \in \ell_1 : \sup_{k \in \mathbb{N}} |x_k| < 1\} \), then for \( x \in U \) we have that

\[
B_1(x, 1 - \|x\|_\infty) = \{y \in \ell_1 : \sum_{k=1}^{\infty} |y_k - x_k| < 1 - \|x\|_\infty\} \subseteq U.
\]

This implies that every open set in \( \ell_\infty \) is open in \( \ell_1 \). Hence, the collections of all Borel measurable sets in \( \ell_1 \) and in \( \ell_\infty \) coincide.

According to the previous arguments, there is an \( S \)-open subset \( A \) of \( \ell_1 \) which is not Borel measurable. Then \( A \) is not Borel measurable in \( \ell_\infty \).

**Theorem 4.3.** For every \( p \in [1, +\infty] \) there exists a strongly separately continuous function \( f : \ell_p \to \mathbb{R} \) such that \( f \notin \bigcup_{\alpha < \omega_1} B_\alpha(\ell_p, \mathbb{R}) \).

**Proof.** Fix \( p \in [1, +\infty] \). By Proposition 4.2 we can find an \( S \)-open subset \( A \subseteq \ell_p \) which is not Borel measurable. For all \( x \in \ell_p \) we put

\[
f(x) = \begin{cases} 
1, & x \in A, \\
0, & x \notin A.
\end{cases}
\]

Notice that \( f \notin \bigcup_{\alpha < \omega_1} B_\alpha(\ell_p, \mathbb{R}) \), since the set \( A = f^{-1}(1) \) is not Borel measurable.

Since \( f(x) = f(y) \) whenever \( y \) differs from \( x \) in at most finitely many coordinates, \( f \) is strongly separately continuous on \( \ell_p \). \( \square \)

5. DISCONTINUITIES OF SSC FUNCTIONS

By \( C(f) \) (\( D(f) \)) we denote the set of all points of continuity (discontinuity) of a map \( f \).

We start with two simple facts.

**Lemma 5.1.** Let \( X \) be a topological space, \( \varphi : X \to \mathbb{R} \) be a continuous function, \( g : X \to \mathbb{R} \) be a bounded function and \( f : X \to \mathbb{R} \) be a function such that \( f(x) = \varphi(x) \cdot g(x) \) for all \( x \in X \). Then \( \varphi^{-1}(0) \subseteq C(f) \).

**Proof.** Fix \( x_0 \in \varphi^{-1}(0) \) and \( \varepsilon > 0 \). Let \( C > 0 \) be a real number such that \( |g(x)| \leq C \) for all \( x \in X \). Since \( \varphi \) is continuous at \( x_0 \), we can find a neighborhood \( U \) of \( x_0 \) such that \( |\varphi(x)| < \frac{\varepsilon}{C} \) for all \( x \in U \). Then

\[
|f(x) - f(x_0)| = |\varphi(x) \cdot g(x)| < \frac{\varepsilon}{C} \cdot C = \varepsilon
\]

for all \( x \in U \). \( \square \)
Lemma 5.2. For any $p \in [1, +\infty)$ the set

\[ D = \left\{ x = (x_k)_{k=1}^\infty \in \ell_p : \sum_{k=1}^\infty \sqrt{|x_k|} = +\infty \right\} \]

is dense in $\ell_p$.

**Proof.** Fix $p \in [1, +\infty)$, $x \in \ell_p$ and $\delta > 0$. We find $N \in \mathbb{N}$ such that

\[ \sum_{k=N+1}^\infty |x_k|^p < \left( \frac{\delta}{2} \right)^p \quad \text{and} \quad \sum_{k=N+1}^\infty \frac{1}{k^{2p}} < \left( \frac{\delta}{2} \right)^p. \]

Let

\[ y = \left( x_1, \ldots, x_N, \frac{1}{(N+1)^p}, \frac{1}{(N+2)^p}, \ldots \right). \]

Clearly, $y \in D$. Moreover,

\[ \|x - y\|_p \leq \left( \sum_{k=N+1}^{\infty} \left( \frac{1}{k^p} \right)^\frac{1}{p} \right)^{\frac{1}{p}} + \left( \sum_{k=N+1}^{\infty} |x_k|^p \right)^\frac{1}{p} < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \]

Hence, $D$ is dense in $\ell_p$. \hfill \Box

**Theorem 5.3.** For any $p \in [1, +\infty)$ and for any open nonempty set $G \subseteq \ell_p$ there exists a strongly separately continuous function $f : \ell_p \to \mathbb{R}$ such that $D(f) = G$.

**Proof.** Fix $p \in [1, +\infty)$. Let $\emptyset \neq G \subseteq \ell_p$ be an open set and $F = \ell_p \setminus G$.

For every $x = (x_k)_{k=1}^\infty \in \ell_p$ we put

\[ \varphi(x) = \begin{cases} \min\{d_\infty(x, F), 1\}, & F \neq \emptyset, \\ 1, & F = \emptyset, \end{cases} \]

\[ g(x) = \begin{cases} \exp\left( -\sum_{k=1}^{\infty} \sqrt{|x_k|} \right), & x \in \ell_{1/2}, \\ 1, & \text{otherwise}, \end{cases} \]

and let

\[ f(x) = \varphi(x) \cdot g(x). \]

Then $F \subseteq C(f)$ by Lemma 5.1.

Now we show that $G \subseteq D(f)$. Assume that $x^0 \in G$. Then $f(x^0) > 0$. We put $\varepsilon = \frac{1}{2} f(x^0)$ and take an arbitrary $\delta > 0$. 
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Since the set $D = \{ x \in \ell_p : \sum_{k=1}^{\infty} \sqrt{|x_k|} = +\infty \}$ is dense in $\ell_p$ by
Lemma 5.2, there exists $x = (x_n)_{n \in \mathbb{N}} \in \ell_p$ such that
\[
\|x - x^0\|_p < \frac{\delta}{2} \quad \text{and} \quad x \in D.
\]
Take a number $N$ such that
\[
\sum_{n=1}^{N} \sqrt{|x_n|} > \ln\left(\frac{2}{f(x^0)}\right) \quad \text{and} \quad \sum_{n=N+1}^{\infty} |x_n|^p < \left(\frac{\delta}{2}\right)^p.
\]
We put $y = (x_1, \ldots, x_N, 0, 0, \ldots)$. Then $y \in \ell_{1/2}$ and
\[
\|y - x^0\|_p \leq \|y - x\|_p + \|x - x^0\|_p = \left(\sum_{n=N+1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} + \|x - x^0\|_p < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]
But
\[
f(x^0) - f(y) = f(x^0) - \varphi(y) \cdot \exp\left(-\sum_{n=1}^{N} \sqrt{|x_n|}\right) > f(x^0) - \exp\left(-\sum_{n=1}^{N} \sqrt{|x_n|}\right) > f(x^0) - \frac{f(x^0)}{2} = \varepsilon,
\]
which implies that $f$ is discontinuous at $x^0$. Therefore, $D(f) = G$.

Now we prove that $g$ is strongly separately continuous. Fix $x^0 \in \ell_p$, $k \in \mathbb{N}$ and $\varepsilon \in (0, 1)$. Take $x = (x_k)_{k=1}^{\infty} \in B_p(x^0, \delta)$ and $y = (x_1, \ldots, x_{k-1}, x^0_k, x_k, \ldots) \in B_p(x^0, \delta)$. If $x \notin \ell_{1/2}$, then $y \notin \ell_{1/2}$. In this case $|g(x) - g(y)| = 0 < \varepsilon$. Assume that $x \in \ell_{1/2}$. Then $y \in \ell_{1/2}$ and
\[
|g(x) - g(y)| = \left| \exp\left(-\sum_{n=1}^{\infty} \sqrt{|x_n|}\right) - \exp\left(-\sum_{n=1}^{\infty} \sqrt{|y_n|}\right) \right| < \left| \exp\left(\sum_{n=1}^{\infty} \left(\sqrt{|y_n|} - \sqrt{|x_n|}\right)\right) - 1 \right| = \exp\left(\sqrt{|x^0_k|} - \sqrt{|x_k|}\right) - 1.
\]
It follows that
\[
|g(x) - g(y)| = \exp\left(\sqrt{|x_k|} - \sqrt{|x^0_k|}\right) - 1 < \exp(\sqrt{\delta}) - 1 = \varepsilon.
\]
in the case $\sqrt{|x^0_k|} - \sqrt{|x_k|} \geq 0$, or
\[
|g(x) - g(y)| < 1 - \exp(-\sqrt{\delta}) < \varepsilon,
\]
on otherwise. Hence, $g$ is strongly separately continuous at $x^0$ with respect to the $k$’th variable.

Finally, $f$ is strongly separately continuous on $\ell_p$ as a product of two ssc functions (see Theorem 3 from [2]).

In connection with Example 2.3 and Theorem 5.3 the following question is natural and open.

**Question 5.4.** Let $G \subseteq \ell_\infty$ be an open nonempty set. Does there exist a strongly separately continuous function $f : \ell_\infty \to \mathbb{R}$ such that $D(f) = G$?

**References**


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